Mathematics 251: Lab 3 MAXIMUM AND MINIMUM VALUES

Please turn in only the printout of your Maple worksheet. Use the **text** feature of Maple to add a header containing your name. Use the **title** option in all plots to introduce a label that will be kept with the plot when your report is printed.

The worksheet in the *seed file* is divided into **Sections** corresponding to the parts of this project description. It also contains almost all you need for problems 0 and 1; and imitating those instructions should allow you to complete the other parts. You may elaborate on this organization in preparing your report. Also, remove from the worksheet any extraneous material and any errors you have made.

In this lab, we use *Maple* to help visualize and compute the maximum and minimum values of a function of two variables. The functions appearing in this lab, like those appearing elsewhere in the course, are usually polynomials, with a few appearances of exponential or trigonometric functions. The names of these standard functions and the usual notation of algebra allow you to write the expressions for their values to which you apply the rules of calculus. The functions will be described here in standard mathematical notation. You will need to supply the translation into Maple idioms. For example, e^{2x^2} must be written as `exp(2*x^2)` or `exp(2*x*x)`.

0. As in Lab 2, we begin by loading the `plots` library and fixing some options. Note that, in contrast to earlier work, we replace the `scaling=CONSTRAINED` option with `scaling=UNCONSTRAINED` since it is not necessary to compare distances along different axes (this is the default setting, so it could be omitted, but it is included to restore this setting if it had been changed). Then we introduce the expressions

$$\frac{y^3}{9} + 3x^2y + 9x^2 + y^2 + xy + 9 \tag{A}$$

$$y(1 - 10xy)e^{-x^2-y^2} \tag{B}$$

that will be studied in this project (you should recognize (B) from problem 10 of Lab 0). Although the discussion of this topic in Calculus uses *functions*, it is easier to work with *expressions* in Maple. Throughout this worksheet, x and y will be treated as independent variables. All other names will stand for expressions depending on those variables. That is, they are the *values* of certain functions of x and y. Thus, we use the name `A` for the expression in (A). We also introduce the region $\mathcal{R}$ for the elliptical disk $9x^2 + (y + 2)^2 \le 9$, and `bR` for an expression that is zero on boundary of $\mathcal{R}$, positive outside R, and negative inside R. These quantities will be used in part 3 (Note the use of the name `bx` for an intermediate expression to simplify the description.) The description of $\mathcal{R}$ is based on expanding to obtain $9x^2 \le 5 + 4y - y^2 = (5 + y)(1 - y)$ and solving for x. The partial derivatives of A and B are also found here. These derivative are needed often in this project, so they need names, and `Ax` is a convenient abbreviation for $\partial A/\partial x$.

```
with(plots):
setoptions3d(axes=BOXED,scaling=UNCONSTRAINED,style=PATCH);
A:=y^3/9+3*x^2*y+9*x^2+y^2+x*y+9;
B:=y*(1-10*x*y)*exp(-x^2-y^2);
bx:=sqrt((5+y)*(1-y))/3
R:=y = -5 .. 1, x = -bx .. bx;
bR:=9*x^2+(y+2)^2-9;
Ax:=diff(A,x); Ay:=diff(A,y);Bx:=diff(B,x); By:=diff(B,y);
```

More. . .

1. In this problem we wish to find and classify all the critical points of the expression (A).

(a) First obtain a rough idea of what this function looks like, by plotting it over the region $-1.5 \le x \le 1.5$, $-7 \le y \le 1$. You should be able to construct the graph without any hints, but be sure to use the `title` option to label it. Examine the graph for possible critical points, but keep your observations to yourself. The results that you calculate later may turn out to be different from what you expect.

(b) Now use Maple's `solve` command to find all the critical points of the expression (A) using the instruction `solA:=solve({Ax,Ay},{x,y});` Note that, if you ask *Maple* to solve an *expression* or a set of expressions, it assumes that you want to solve the equations in which all expressions are equal to zero. It is also a good idea is to name the solutions when they are computed, so we assign the result the name `solA`. The solutions obtained from the `solve` command are given to you as an *expression sequence*, i.e., several quantities separated by commas (as in the definition of $\mathcal{R}$ above). In this case, the elements in the sequence are sets of assignments of the variables. Then you can retrieve individual solutions using the usual *Maple* indexing convention, e.g., `solA[3]` refers to the third one. To check this, you can evaluate both `Ax` and `Ay` at `solA[3]` by typing `eval([Ax,Ay],solA[3]);`. The brackets signify a *list* of quantities (the same structure that we used to represent vectors in Lab 1), and the substituted values will appear in the same order in the output as the expressions did in the input. The "seed file" contains this example, with the warning that it should be removed after you try it. Before you do, make sure you understand why the result is *not* surprising. (In earlier versions, the `subs()` instruction was used. The order in which the arguments appear differs, but the results are similar. In general, `eval()` is more useful for our applications, so lab descriptions and seed files are being changed to encourage its use.)

(c) Again using Maple's `diff` command, evaluate the second derivatives of (A). The names `Axx`, `Axy`, `Ayx`, and `Ayy` are reasonable choices. Using this convention, `Axy` and `Ayx` are obtained by different computations, but they should be equal. The textbook (see p. 940) classifies critical points using a second derivative test involving a quantity

$$D = \frac{\partial^2 A}{\partial x^2} \cdot \frac{\partial^2 A}{\partial y^2} - \frac{\partial^2 A}{\partial x\, \partial y} \cdot \frac{\partial^2 A}{\partial y\, \partial x}.$$

You are encouraged to **try to** call this expression D. It will lead to an error message, and you can find an explanation of that message using the help facility. When you find it, **add a comment to your worksheet**. Except for this comment, **remove all evidence of the error from the final worksheet**, and **compute this expression** D (using a different name) and **evaluate it**, together with **anything else that you need to classify the critical point**, at **each** critical point. Your report should also include **text** describing the information given by this second derivative test. When you have these values, add a line or two of text giving the type of each critical point. (You should be able to this without any hints, so the seed file contains only a few blank lines in this section.)

More. . .

2. Now consider the expression B. If you don't have the graph from Lab 0 handy, recreate it, but don't include it in your report. You can use a separate "scratch" worksheet for things that you don't want to include in your report. When using the **shared kernel mode** (the default setting before Maple 9), all definitions made during the session would be known on that sheet. (This works the other way also, so be careful. Something done on your scratch sheet may introduce a change that affects your main sheet. Your report is expected to be consistent, so it should be checked by selecting "Execute Worksheet" from the Edit menu in a fresh session.) In Maple 9 you will be in **parallel kernel mode** unless you insist otherwise. In this mode, a scratch worksheet will neither affect your main worksheet nor share definitions made there. Here, we are just producing a graph without introducing any definitions, so both modes should have the same behavior, although you will need to copy some definitions to produce the graph.

(a) First, use the `solve` command to find critical points as in problem 1. An instruction is included in the seed file to do this and assign the name `solB` to the result. Space is then left for you to insert a **text** description of your interpretation of the result. *Maple help* should be used to identify any unfamiliar expressions. Then, the instruction `solBv:=evalf(allvalues(solB));` is included to get numerical values of these quantities. Again, space is then left for you to insert a **text** description of your interpretation of the result. Although we emphasize a different method of finding the critical points in the remainder of this problem, this use of `allvalues` will appear again in problem **3c**.

(b) An alternative is to use the `fsolve` command to find the critical points that lie in the region $-2 \leq x \leq 2$, $-2 \leq y \leq 2$. There are exactly four of them, as indicated by the plot from Lab 0. You should also find the values of the function the expression (B) at all of these critical points; be sure that your worksheet makes it clear which values are obtained at which points. The `fsolve` command uses an iterative method (such as Newton's method) to find the roots and sometimes the method does not converge to a root. You should consult the help file for the `fsolve` command to find how to restrict the search for a root to a smaller region, and use your plot to identify suitable regions. The seed file contains the line `p1:= fsolve({Bx,By},{x,y}, x =-1..0, y=0..1);` that finds one of the critical points and assigns it the name `p1`. An efficient way to evaluate B at this point using `eval` is also included in the seed file. Use this as a model for finding the value of B at all critical points.

(c) This function is close to zero if $x^2 + y^2$ is large, and the plot reveals that it takes both positive and negative values. The maximum and minimum must be attained at critical points, and you should now know, and have names for, all of them to reasonable accuracy. Use this to determine the absolute minimum and absolute maximum values of the expression (B). Summarize in **text**.

More. . .

3. In this problem, we find the absolute minimum and absolute maximum of the expression (A) of Problem 1 on the ellipse $\mathcal{R}$ whose description was given at the beginning of this Lab (and included in the seed file). We know from the general theory that the absolute minimum and maximum of the expression A occur either (1) at critical points of the expression A which lie in the interior of the region $\mathcal{R}$ or (2) on the boundary of the region $\mathcal{R}$. The boundary is a smooth arc on which the method of *Lagrange Multipliers* applies.

(a) Obtain a plot of the expression A with the domain restricted to the region $\mathcal{R}$. The seed file contains an instruction that will produce the plot and supply a title. You should adjust the view to obtain a plot that will guide the determination of the extreme values and the points at which they are attained.

(b) Determine which of the critical points found in Problem 1 lie in the region $\mathcal{R}$ and evaluate the expression A at these points. The easiest way to do this is to evaluate $bR = 9x^2 + (y+2)^2 - 9$ at the values named by each `solA[i]`. Those points giving negative values are inside the ellipse. An example of this is included in the seed file. State your conclusions in a **text** section, but leave the calculations supporting your conclusion in the worksheet. (An alternative would be follow the definition of R by substituting the y value at each point into `bx` and determine whether the corresponding x value lies between this quantity and its negative. You may use this to check your conclusions, but do not leave such explorations in the final worksheet. Only the testing of $9x^2 + (y+2)^2 - 9$ at critical points should be shown.)

(c) For functions of two variables, the criterion for a extreme value of $f(x, y)$ subject to the constraint $g(x, y) = 0$ specializes to $f_x g_y = f_y g_x$ when the *Lagrange multiplier* is eliminated. Solving this simultaneously with $g(x, y) = 0$ determines the points that must be considered. The seed file contains the instructions

```
Ltest:=simplify(Ax*diff(bR,y)-Ay*diff(bR,x));
BCpts:=solve({Ltest,bR},{x,y});
```

(note the use of `simplify` in the definition of `Ltest` to allow human observation of the elimination of the Lagrange multiplier). The result of these two instructions is an *algebraic* solution that is not immediately useful for our purpose. However, the method used in problem **2a** will yield a list of all *complex number* solutions of the equation.

Create this list and select the *real number* solutions in it. Then, evaluate A at these points.

(d) Combine the results of (b) and (c). You now have a list of all possible locations of maxima and minima on $\mathcal{R}$ (both interior points and boundary points) and the value of A at these points. By identifying the smallest and largest values of A in this list, you will have found the extreme values on $\mathcal{R}$ and point where those values are attained. You will **not** need the classification of critical points from problem **1**, although the results should be *consistent* (i.e. a *global* maximum or minimum at an interior point must be a local extremum of the same type).

Use **text** to state your conclusions.

End of Lab 3