Mathematics 251 Maple Lab 2

Quadric Surfaces

Fall 2004

This project The worksheet in the **seed file** contains instructions to guide you through one example, with space left in other **Sections** for you to fill in the modified instructions needed for the other examples. Space is left at the top of the worksheet for you to use the **text** feature of Maple to add a header containing your name and any other information requested by your instructor. This worksheet should be printed and submitted for grading.

This project contains three dimensional graphs. Interactive tools for modifying those graphs make heavy use of computer resources and do not work well over a network. However, there should be no difficulty in any of the University labs or a personal copy of Maple on your computer.

Another concern is that Maple will stop responding in the middle of a session, so you should **save your work often**. You may also enable *Auto-Save* using *Options* under the *Tools* menu. The same notebook allows you to change the **kernel mode**. Starting with Maple9, the standard is **parallel kernel**. This mode is assumed when suggesting use of a **supplementary worksheet** that will not affect your main worksheet. If you have unexpected results, check the kernel mode and return it to the standard setting if necessary.

It is also possible to do the different projects on separate worksheets with the results combined after they are printed. If you do this in parallel kernel mode, you will need to include the definitions from Section 0 in each worksheet.

A quadric surface is the graph of a second-degree equation in three variables x, y and z. The most general form of such an equation is:

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0,$$

where A, B, C, D, E, F, G, H, I, and J are constants. In this lab, we use Maple to help visualize some types of quadric surfaces that can arise from equations of the above form.

There are three Maple commands that we will use to plot surfaces in three dimensions. These are `plot3d`, `implicitplot3d`, and `display`. The first is always available, but before the second or third of these can be used, one must type the command `with(plots):`. The command `plot3d` is used to **graph of a function**, this is, a surface when one of the variables can be solved for explicitly in terms of the other two. It can also be used for **parametric** surfaces in which x, y and z are given as functions of u and v (or any other names that you prefer). In all cases, ranges of independent variables should be **specified**, but the endpoints **need not** all be **constant**. The rule is that the **first variable** must have limits that **evaluate to constants** (in addition to explicit constants, you may say `x=-a..a` when specifying this part of the range if `a` has been assigned a constant value by an instruction like `a:=3;`), but the limits for the **second variable** are **allowed to depend on the first variable**.

The command `implicitplot3d` is used to graph any surface of the form $g(x, y, z) = 0$. This instruction requires **constant** limits for the three variables x, y and z. (Again, these constants are not required to be explicit — an expression that evaluates to a constant may be used.)

Because the implementation is different, graphs of the same surface drawn using different instructions may not look the same. One theme in this project examines the differences between graphs that are the same **in theory**. Sometimes the difference is minor, but you should notice **all** those differences between plots caused by different implementations.

In this lab, plot commands should be used to **create named plots**. These commands should **always** end with a colon: the output it generates is not particularly useful, although you may want to see it **once** (such excursions are best confined to the **supplementary worksheet**). Current versions of Maple show information about the plot structure that may be useful in diagnosing errors, but versions prior to Maple8 would fill your worksheet with a list of points that were to be plotted. After defining a named plot, if you enter the name of the plot, followed by a semicolon, the plot will appear. The main reason for naming plots is to allow them to be combined with the `display` command. This command is used to combine previously defined plots at the same time on the same set of axes. Additional options, like `title` can be used with the `display` command.

You may mix plots formed with `plot3d` and `implicitplot3d` in the same `display`, but you may not mix two dimensional plots and three dimensional plots.

Part 0: Introduction

Some Maple commands are introduced. In addition to loading the `plots` library, some standard options for all plots are fixed. These assure that a plot will include coordinate axes (framing the graph) and that distances on all axes will be the same. Symbolic names are also introduced for constants that will appear in all parts of this project.

In this section, you are mostly only required to observe, but part (d) asks for an interpretation of the graphs that were drawn.

(a) Execute the following sequence of commands from the seed file. (The `plots` library will be used throughout the project, but the `VectorCalculus` library is only need in the Part 2).

```
with(plots):with(VectorCalculus):
setoptions3d(axes=BOXED,scaling=CONSTRAINED,style=PATCH);
SetCoordinates( 'cartesian'[x,y,z] );
a:=3;b:=4;c:=6;
p1:=plot3d(a, y=0..b,z=0..c, color=RED):
p2:=plot3d(c, x=0..a,y=0..b*x/a, color=BLUE):
display({p1,p2},title="Planes using simple plot3d");
```

The colors won't appear when printed in black and white, so be sure to identify how different parts were colored when you describe the plots in your discussion in part (d).

(b) Execute these commands from the seed file.

```
p3:=plot3d([a,y,z], y=0..b,z=0..c, color=RED):
p4:=plot3d([x,y,c], x=0..a,y=0..b*x/a, color=BLUE):
display({p3,p4},title="Planes using parametric plot3d");
```

(c) Execute these commands from the seed file.

```
p5:=implicitplot3d(x=a, x=0..a, y=0..b,z=0..c, color=RED):
p6:=implicitplot3d(z=c, x=0..a,y=0..b,z=0..c, color=BLUE):
display({p5,p6},title="Planes using implicitplot3d");
```

(d) Although it **looks like** all parts are asking to plot parts of the planes $x = a$ and $z = c$, where $a = 3$ and $c = 6$ because these names were fixed at the start of the worksheet. the results look different. Answer in **text**: which graphs give the intended results?

Also, only constant limits can be used in implicit plots, so the triangular region used in the other parts was not available in part (c). Answer the following questions in **text**. Are there any **other differences** that you notice between the plots? Which method do you prefer for producing these graph? Why do you prefer that method?

Part 1: An ellipsoid

A quadric surface of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{1}$$

where a, b, and c are positive constants is an ellipsoid. In this problem, we use this general formula, but we have fixed $a = 3$, $b = 4$, $c = 6$ previously, so we get

$$\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{36} = 1.$$

(a) One approach to plotting the surface given by (1) is to solve for z, obtaining the two surfaces

$$z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}, \qquad z = -c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

which correspond to the top and bottom half of the ellipsoid. Each of these surfaces is defined explicitly in the form $z = f(x, y)$ and thus can be graphed using the `plot3d` command. The `plot3d` command requires you to specify a domain over which you wish to plot the graph. For example, `top:=c*sqrt(1-x^2/a^2-y^2/b^2);` is an expression for the top half of the ellipsoid, and `plot3d(top,x=-a..a,y=-b..b);` is a request to plot its graph over a rectangular region containing all points for which the expression `top` can be defined. The **supplementary worksheet** contains this version of the graph to allow comparison with the graph over the **exact domain**, described below, that is the only graph constructed in the seed file.

It is also possible to construct plots over domains that are not rectangles. The exact domain for which the expression `top` yields real values is

$$-a \le x \le a; \qquad -b\sqrt{1-(x^2/a^2)} \le y \le b\sqrt{1-(x^2/a^2)}.$$

The seed file contains instructions for obtaining this plot.

(b) The top and bottom of the ellipsoid can be plotted together in the same `plot3d` command. The whole ellipsoid can also be plotted using the `implicitplot3d` command. The seed file contains instructions for constructing plots in both styles, although the actual plot in the implicit case is left for you. Note that the name $E1$ is introduced for the implicit plot to allow it to be used in a `display` command. A title may be added in that command.

(c) A parametric description of the ellipsoid can be given using a variant on the **polar coordinate** description of the surface of a sphere. The seed file contains instructions to construct a plot called $E2$ of our ellipsoid. After executing this instruction, you should use a `display` command to add a title to this graph.

(d) Use the instructions in the **supplementary worksheet** to find the intersection of the ellipsoid $E1$ with the planes $x = 1$ and $y = 2$. Rotate the plot until it clearly shows the curve of intersection. Make note of the values of θ and ϕ on the **context bar**. Then write a command to draw this plot with those values in an `orientation` option as in Lab 1 and copy it into your main worksheet.

(e) Here are some items for **discussion**.

First, compare the surfaces $E1$ and $E2$. Both show the whole ellipsoid, you should notice differences in the style of presentation of the surface. Give a **text** description of differences that you can identify.

Second, again in **text** describe the intersections with planes that you have plotted. In particular, consider the following questions.

What is the intersection of the ellipsoid $E1$ with the plane $x = 1$, and the intersection of the ellipsoid $E1$ with the plane $y = 2$. Do the algebra, either by hand, or with the aid of Maple, to find an equation for the intersection, and use this equation to identify the curve. Use the **text** feature of Maple to insert a verbal description of these curves (e.g., the **name** of this type of curve and the **location** of **key points** on the curve). Does the result of this algebra agree with the plots obtained by Maple?

Part 2: An elliptic hyperboloid

The surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad (2)$$

is known as an (elliptic) **hyperboloid of one sheet**. Continue to use the values $a = 3$, $b = 4$, $c = 6$. This time, we see from (2) that the surface can only be defined for (x, y) **outside** the ellipse $x^2/a^2 + y^2/b^2 = 1$, so a larger domain is needed in order to get a good view of the surface, so we choose

$$-2a \leq x \leq 2a, \quad -2b \leq y \leq 2b, \quad -2c \leq z \leq 2c. \qquad (2a)$$

(a) In both your **main worksheet** and the **supplementary worksheet**, use `implicitplot3d` to obtain a name for a graph of the surface with equation (2) with the bounds given by $(2a)$. In the supplementary worksheet, you can enter that name followed by a semicolon to show this surface, but only the `display` described later should appear in the main worksheet..

(b) Construct the tangent plane to (2) at $(3, 4, 6)$. You may want to verify (in the **supplementary worksheet**) that this point lies on the surface (2) by **evaluating** the equation at this point. If `sur2` has been assigned a value that is the equation of the surface, then `eval(sur2,{x=a,y=b,z=c});` will return a **true equation**. Also, if `sur2` has been written with all terms containing variables on its left side, and `pt2` represents the `Vector` $\langle a, b, c\rangle$, then `dir2 := evalVF(Gradient(lhs(sur2)),pt2);` will produce a `Vector` perpendicular to the surface at this point (the `evalVF` command is necessary to get a result that will be treated as a vector when you use a **dot product** to find the equation of the plane.

If you have difficulty getting Maple to produce the tangent plane, you may find its equation by hand.

Produce a plot containing the hyperboloid and this tangent plane **with each surface drawn in a solid color** (you can choose the color). You may also want to use the option `style=PATCHNOGRID` to hide the grid lines. Find a view that shows the shape of this intersection, record these values of θ and ϕ and use them in the `orientation` option of the plot that you include in the main worksheet.

(c) Describe **in text** the intersection of this surface with its tangent plane.

End of Lab 2