Mathematics 251 Maple Lab 3

Maximum and Minimum Values

Fall 2004

This project

The worksheet in the **seed file** is divided into **Sections** corresponding to the parts of this project description. It also contains almost all you need for problems 0 and 1; and imitating those instructions should allow you to complete the other parts. Your main worksheet will grow from this file. You should remove any extraneous material from the worksheet, but you may elaborate on the organization in preparing your report. There is also a **supplementary worksheet** that contains instructions to refine the Maple commands used in the main worksheet. Please turn in only the printout of your **main Maple worksheet**. Use the **text** feature of Maple to add a header containing your name and any other information requested by your instructor. Use the **title** option in all plots to introduce a label that will be kept with the plot when your report is printed.

In this lab, we use **Maple** to help visualize and compute the maximum and minimum values of a function of two variables. The functions appearing in this lab, like those appearing elsewhere in the course, are usually polynomials, with a few appearances of exponential or trigonometric functions. The names of these standard functions and the usual notation of algebra allow you to write the expressions for their values to which you apply the rules of calculus. The functions will be described here in standard mathematical notation. You will need to supply the translation into Maple idioms. For example, e^{2x^2} must be written as `exp(2*x^2)` or `exp(2*x*x)`.

0. Setup

As in Lab 2, we begin by loading the `plots` and `VectorCalculus` libraries and fixing some options. Unfortunately, while the `VectorCalculus` package builds matrices, it doesn't allow easy access to their determinants, which are often more useful for this course. This omission is fixed by introducing the name `DetM` for the determinant function from the `LinearAlgebra` package. We introduce the expressions

$$\frac{y^3}{9} + 3x^2y + 9x^2 + y^2 + xy + 9 \qquad \text{(A)}$$

$$y(1 - 8xy)e^{-2x^2-y^2} \qquad \text{(B)}$$

that will be studied in this project (you should recognize (B) from problem 7 of Lab 0). Although the discussion of this topic in Calculus uses **functions**, it is easier to work with **expressions** in Maple. Throughout this worksheet, x and y will be treated as independent variables. All other names will stand for expressions depending on those variables. That is, they are the **values** of certain functions of x and y. Thus, we use the name `A` for the expression in (A). We also introduce the region $\mathcal{R}$ for the elliptical disk $9x^2 + (y+2)^2 \leq 9$, and `bR` for an expression that is zero on boundary of $\mathcal{R}$, positive outside $\mathcal{R}$, and negative inside $\mathcal{R}$. These quantities will be used in part 3 (Note the use of the name `bx` for an intermediate expression to simplify the description.) The description of $\mathcal{R}$ is based on expanding to obtain $9x^2 \leq 5 + 4y - y^2 = (5+y)(1-y)$ and solving for x. The gradients of A and B are also found here. Instructions for recovering individual partial derivatives from the gradient are given in the **supplementary worksheet**. We won't repeat the contents of the worksheets here.

1. Local extrema

In this problem we wish to find and classify all the critical points of the expression (A).

(a) First obtain a rough idea of what this function looks like, by plotting it over the region $-1.5 \le x \le 1.5$, $-7 \le y \le 1$ in the **supplementary worksheet**. You should be able to construct the graph without any hints. Rotate the plot until it gives a good view of the critical points. Make note of the values of θ and ϕ on the **context bar**. Then write a command to draw this plot with those values in an `orientation` option as in previous labs, add a suitable `title`, and copy it into your main worksheet.

(b) Now use Maple's `solve` command to find all the critical points of the expression (A) using the instruction `solA:=[solve(convert(GA,set))];`. Note that, if you ask **Maple** to solve an **expression** or a set of expressions, it assumes that you want to solve the equations in which all expressions are equal to zero. Recent versions of Maple allow you to omit the **variables that you are solving for** if you are solving for **all variables in the expression**. The solutions obtained from the `solve` command are given to you as an **expression sequence**, i.e., several quantities separated by commas (as in the definition of $\mathcal{R}$ above). In this case, the elements in the sequence are sets of assignments of the variables. Then you can retrieve individual solutions using the usual **Maple** indexing convention, e.g., `solA[3]` refers to the third one. This is good enough for what we will be doing with the solutions, but Maple recommends enclosing the output of `solve` in brackets, so that it becomes the more versatile **list**. Finally, it is also a good idea is to name the solutions when they are computed, so we assign the result the name `solA`. In the **supplementary worksheet**, the solutions are checked with the command `eval(GA,solA[2]);`. Be sure that you recognize that this is a **check** of the second solution. You can check the other solutions in the same way.

(c) The `VectorCalculus` package has the function `Hessian` that computes the **second partial derivatives** and **arranges them in a matrix**. The **supplementary worksheet** identifies the individual partial derivatives in this matrix.

The **equality of mixed second partial derivatives** shows that the Hessian is always a **symmetric matrix**, i.e., it is unchanged when reflected in its main diagonal.

The textbook classifies critical points using a second derivative test involving a quantity

$$D = \frac{\partial^2 A}{\partial x^2} \cdot \frac{\partial^2 A}{\partial y^2} - \frac{\partial^2 A}{\partial x\, \partial y} \cdot \frac{\partial^2 A}{\partial y\, \partial x}.$$

This is just the **determinant** of the Hessian matrix.

For **each critical point**, apply the `DetM` function to the result of **evaluating** the `Hessian` matrix at that point. Include other results needed to **classify** the critical point using the second derivative test described in the textbook. The calculations concerning each point should be **summarized** by a **text statement** identifying the **type of critical point**. Include the textbook's characterization of the test for that type of point together with the results of the calculations that support it.

2. Global extrema

Now consider the expression B. If you don't have the graph from Lab 0 handy, recreate it in the **supplementary worksheet**.

(a) First, use the `solve` command to find critical points as in problem 1 and save the result under the name `solB`. Space is then left for you to insert a **text** description of your interpretation of the result. **Maple help** should be used to identify any unfamiliar expressions. Repeat this in the **supplementary worksheet** and then write `allvalues(solB);` to examine the individual roots. These should not be copied to the main worksheet, but space is left in the seed file for you to insert a **text** description of your **interpretation** of the result and **comparison** with the use of `allvalues` alone. In particular, if the solutions are given as roots of a **polynomial**, one expects that the number of roots should equal the **degree** of the polynomial. Also, if the coefficients of the polynomial are real, any complex roots should occur in **conjugate pairs**. Your discussion should mention the number of **real solutions** that you expect, and evidence that some of the solutions found by Maple may be real although they may be written with a nonzero imaginary part.

Although we emphasize a different method of finding the critical points in the remainder of this problem, this use of `allvalues` will appear again in problem **3c**.

(b) An alternative is to use the `fsolve` command to find the critical points of B that lie in the region $-2 \le x \le 2$, $-2 \le y \le 2$. There are exactly four of them, as indicated by the plot from Lab 0. You should also find the values of the function the expression (B) at all of these critical points; be sure that your worksheet makes it clear which values are obtained at which points. The `fsolve` command uses an iterative method (such as Newton's method) to find the roots and sometimes the method does not converge to a root. You should consult the help file for the `fsolve` command to find how to restrict the search for a root to a smaller region, and use your plot to identify suitable regions. The seed file contains the line `p1:=fsolve(convert(GB,set),{x =-1..0, y=0..1});` that finds one of the critical points and assigns it the name `p1`. You can then evaluate B at this point using `eval(B,p1);` . Use this as a model for finding the value of B at all critical points.

(c) This function is close to zero if $x^2 + y^2$ is large, and the plot reveals that it takes both positive and negative values. The maximum and minimum must be attained at critical points, and you should now know, and have names for, all of them to reasonable accuracy. Thus, it is not necessary to distinguish the **local maxima** when searching for a **global maximum**. The **global maximum** is simply the **largest value** of the function at a critical point. Use this method to determine the absolute minimum and absolute maximum values of the expression (B). Summarize in **text**.

3. Extrema in a region.

In this problem, we find the absolute minimum and absolute maximum of the expression (A) of Problem 1 on the ellipse $\mathcal{R}$ whose description was given at the beginning of this Lab (and included in the seed file). We know from the general theory that the absolute minimum and maximum of the expression A occur either (1) at critical points of the expression A which lie in the interior of the region $\mathcal{R}$ or (2) on the boundary of the region $\mathcal{R}$. The boundary is a smooth arc on which the method of **Lagrange Multipliers** applies.

(a) Obtain a plot of the expression A with the domain restricted to the region $\mathcal{R}$. The **supplementary worksheet** contains an instruction that will produce the plot. You should adjust the view to obtain a plot that will guide the determination of the extreme values and the points at which they are attained and use this to select θ and ϕ in an `orientation` option. Test the command again with this option and a `title`, and copy the resulting `plot` command to your **main worksheet**.

(b) Determine which of the critical points found in Problem 1 lie in the region $\mathcal{R}$ and evaluate the expression A at these points. The easiest way to do this is to evaluate $bR = 9x^2 + (y+2)^2 - 9$ at the values named by each `solA[i]`. Those points giving negative values are inside the ellipse. An example of this is included in the seed file. Leave these calculations for all critical points of A in the worksheet and identify the critical points belonging to $\mathcal{R}$ in a **text** statement. Since the **selection** of critical points inside $\mathcal{R}$ is essential to building a list of points to consider, you should be careful with this test. A large value outside $\mathcal{R}$ is irrelevant, and failing to include an interior maximum point can lead to results that appear inconsistent. Other tests may be performed in the supplementary worksheet (one alternative would be to mimic the definition of $\mathcal{R}$ by substituting the y value at each point into `bx` and determine whether the corresponding x value lies between this quantity and its negative).

(c) For functions of two variables, the criterion for a extreme value of $f(x, y)$ subject to the constraint $g(x, y) = 0$ specializes to $f_x g_y = f_y g_x$ when the **Lagrange multiplier** is eliminated. This is equivalent the matrix whose columns are the gradients of f and g having determinant zero. Solving this equation simultaneously with $g(x, y) = 0$ determines the points that must be considered. Both the seed file and supplementary worksheet contain the instructions

```
GAv:=convert(GA,Vector);
GbR:=Gradient(bR);
GbRv:=convert(GbR,Vector);
Ltest:=DetM(<GAv|GbRv>);
```

This is followed, in the supplementary worksheet, by `BCpts:=solve(Ltest,bR;` . The result of these instructions is an **algebraic** solution that is not immediately useful for our purpose. However, the **supplementary worksheet** contains various tools, similar to those used in part **2a** to list of all **complex number** solutions of the equation and select the solutions in **real numbers**.

Return to the main worksheet and give a **text report** on the number of complex number solutions that you found and the number of these that are real number solutions. Then, supply an **efficient** method for constructing these points. For example, if you have isolated the solutions into distinct sets, the method of part **2b** may be used.

(d) Combine the results of (b) and (c). That is, form a list of all possible locations of maxima and minima on $\mathcal{R}$ consisting of the **interior** critical points of A and the boundary points satisfying the **Lagrange multiplier** criterion and the value of A at these points. By identifying the smallest and largest values of A in this list, you will have found the extreme values on $\mathcal{R}$ and point where those values are attained. You will **not** need the classification of critical points from problem **1**, although the results should be **consistent** (i.e. a **global** maximum or minimum at an interior point must be a local extremum of the same type).

Use **text** to state your conclusions about the largest and smallest values of the function A on the set $\mathcal{R}$. Be sure that your answers here are consistent with your graph.

End of Lab 3