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LAB 4: Vector Spaces and General Solution to $A\mathbf{x} = \mathbf{b}$

In this lab you will use MATLAB to study the following topics:

- The subspace spanned by a set of vectors, and the fundamental concepts of *independence, basis* and *dimension.*
- Two important subspaces associated with an $m \times n$ matrix A: the *column space* of A in $\mathbf{R}^m$ of dimension = rank(A), and the *null space* of A in $\mathbf{R}^n$ of dimension = nullity(A).
- The complete solution of the linear equation $A\mathbf{x} = \mathbf{b}$, where A is a given $m \times n$ matrix, $\mathbf{b}$ is a given $m \times 1$ column vector and $\mathbf{x}$ is an unknown $n \times 1$ column vector.

Reading from Textbook: The linear algebra ideas in this lab are found in Sections 6.1 through 6.6 of the text. You should read the text and work the suggested problems for each section is parallel with working on this lab.

Tcodes: For this lab you will need the Teaching Codes

`nulbasis.m, elim.m, partic.m, plot2d.m, house.m`

(the last two are only used in the extra-credit question). Before beginning work on the Lab questions you should copy these codes from the Teaching Codes directory on the Math Department/Course Materials/Linear Algebra 250C web page to your diskette (see Lab 3 for more details).

Script files: You will need the MATLAB script files `rvect.m` and `rmat.m` from Lab 3 (if you didn't do Lab 3, get a copy of that assignment and follow the directions there to create these m-files). Be sure that you have set the path in MATLAB so that MATLAB can find your own m-files and the Teaching Codes.

Lab Write-up: You should open a diary file at the beginning of each MATLAB session (see Lab 1 for details). Insert comments in your diary file as you work through the assignment. Be sure to answer all the questions in the lab assignment.

Random Seed: When you start your MATLAB session, initialize the random number generator by typing

```
rand('seed', abcd)
```

where *abcd* are the last four digits of your Student ID number. This will ensure that you generate your own particular random vectors and matrices.

BE SURE TO INCLUDE THIS LINE IN YOUR LAB WRITE-UP

The lab report that you hand in must be your own work. The following problems all use randomly generated matrices and vectors, so the matrices and vectors in your lab report will not be the same as those of other students doing the lab. Sharing of lab report files is not allowed in this course.

Question 1. Independence, Basis and Dimension

Generate four random vectors in $\mathbf{R}^3$ by the command

```
x = rvect(3), y = rvect(3), z = rvect(3), w = rvect(3)
```

Use these vectors in the following.

(a): Is the set $\{\mathbf{x}, \mathbf{y}\}$ linearly independent? Why? Answer this first without MATLAB, with a hand calculation. Then form the matrix A with the vectors $\mathbf{x}, \mathbf{y}$ as columns and calculate its rank.

```
A = [x, y],   r = rank(A)
```

What values for r would show that the set $\{\mathbf{x}, \mathbf{y}\}$ is linearly *dependent*?

Next, take an arbitrary linear combination $\mathbf{u}$ of $\mathbf{x}$ and $\mathbf{y}$ and calculate the rank of the augmented matrix:

```
u = rand(1)*x + rand(1)*y,   s = rank([A, u])
```

Can you conclude from the value obtained for s that u is in the subspace spanned by $\mathbf{x}, \mathbf{y}$? Why?

Finally, take an arbitrary linear combination $\mathbf{v}$ of $\mathbf{x}, \mathbf{z}$ and calculate the rank of the augmented matrix:

```
v = rand(1)*x + rand(1)*z,   t = rank([A, v])
```

Can you conclude from the value obtained for t that $\mathbf{v}$ is *not* in the subspace spanned by $\mathbf{x}, \mathbf{y}$? Why?

(b): Form the matrix B with the vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}$ as columns and calculate its rank.

```
B = [x, y, z, w],   r = rank(B)
```

What can you conclude about the linear independence or dependence of the set of vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}$? Can you make this same conclusion for *every* set of four vectors in $\mathbf{R}^3$? Why?

Question 2. Null space of A

(a) Special Solutions to $A\mathbf{x} = 0$ when A is completely random: The MATLAB Teaching Code `nulbasis.m` calculates the *special solutions* to the equation $A\mathbf{x} = 0$. Try it first with

```
A = rmat(3, 6), R = rref(A), N = nulbasis(A)
```

The columns of N are the *special solutions* to $Ax = 0$, called $\mathbf{x}_1, \ldots, \mathbf{x}_p$ on page 277 of the text ($p = \text{nullity}(A)$). Check that $A * N = 0$ by MATLAB.

Define

```
x1 = N(:,1),   x2 = N(:,2),   x3 = N(:,3)
```

(Notice that `x1` is a whole 6-component *vector*, not a scalar.) Which component of $\mathbf{x}_1$ *must be* 1? Which components of $\mathbf{x}_1$ *must be* 0? Answer the same questions for $\mathbf{x}_2$ and $\mathbf{x}_3$

(b) General Solution to $A\mathbf{x} = 0$ when A is partly random: Generate a *partly random* 3×6 matrix A by

```
B = rmat(3, 2); C = rmat(3,2);
A = [B, 3*B, C], R = rref(A), N = nulbasis(A)
```

Which columns of R contain the pivots? What is the *nullity* of A?

Define the *special solutions* to $A\mathbf{x} = 0$ as in part **(a)**:

```
x1 = N(:,1),   x2 = N(:,2),   x3 = N(:,3)
```

Now generate a random linear combination $\mathbf{x}$ of the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ by

```
x = rand(1)*x1 + rand(1)*x2 + rand(1)*x3
```

(Each occurrence of `rand(1)` generates a different random coefficient). Check by MATLAB that $A\mathbf{x} = 0$. Explain (without MATLAB) why $\mathbf{x}$ also satisfies $R\mathbf{x} = 0$. Then verify this by MATLAB.

Question 3. Solving $A\mathbf{x} = \mathbf{b}$

(a) Particular Solution (underdetermined system): Generate a random 3×5 matrix A (the coefficient matrix for an *underdetermined* system of 3 equations in 5 unknowns) and its reduced row echelon form R by

```
A = rmat(3,5),   R=rref(A)
```

Now generate a random 3×1 vector $\mathbf{b}$ and use the Teaching Code `partic.m` to find a particular solution to $A\mathbf{x} = \mathbf{b}$ by

```
b = rmat(3,1),   x = partic(A, b)
```

Check that $A\mathbf{x} = \mathbf{b}$. Repeat for another random vector $\mathbf{b}$, using the same matrix A. What entries in x are zero both times? Which columns of R correspond to free variables? Calculate `rank(A)` and `rank([A, b])` (the augmented matrix). Does the equation $A\mathbf{x} = \mathbf{b}$ have a solution for *every* vector $\mathbf{b}$ in this case? Why? (See Theorem 6.14 on page 290).

(b) Particular Solution (overdetermined system): Generate a random 5×3 matrix `A = rmat(5,3)` (the coefficient matrix for an *overdetermined* system of 5 equations in 3 variables). The following MATLAB command will generate a random 5×1 vector $\mathbf{b}$ and try to find a particular solution to $A\mathbf{x} = \mathbf{b}$:

```
b = rmat(5,1),   x = partic(A, b)
```

What did MATLAB return for $\mathbf{x}$? Calculate `rank(A)` and `rank([A, b])`. Do you get the same relation between the ranks as you got in part **(a)**? Use this information to explain why there is no solution to $A\mathbf{x} = \mathbf{b}$ for a completely random choice of $\mathbf{b}$. (See Theorem 6.14 on page 290).

Now use the (partly) random vector

```
b = rand(1)*A(:,1) + rand(1)*A(:,2) + rand(1)*A(:,3)
```

and calculate `x = partic(A, b)`. Explain why the special form of the vector $\mathbf{b}$ guarantees that there always is a solution $\mathbf{x}$, no matter what the random coefficients might be. Since there is a solution to $A\mathbf{x} = \mathbf{b}$ for this $\mathbf{b}$, what is the rank of the augmented matrix `[A, b]`? Answer first without calculation (see Theorem 6.14 on page 290), and then check using MATLAB.

(c) General Solution (underdetermined system): Execute the commands

```
A = rmat(3,5), b = rmat(3,1), N = nulbasis(A), xp = partic(A,b)
```

Set `x1 = N(:,1), x2 = N(:,2)` and form a random *general solution*

```
x = xp + rand(1)*x1 + rand(1)*x2
```

to $A\mathbf{x} = \mathbf{b}$. Check by MATLAB that $A\mathbf{x} - \mathbf{b} = 0$.

Now solve the equation $A\mathbf{x} = \mathbf{b}$ with the extra condition that $\mathbf{x}$ should be of the form

$$\mathbf{x} = [x_1,\, x_2,\, x_3,\, -9,\, 8]^{\mathrm{T}}$$

For this, you must choose particular scalars c_1 and c_2 so that

$$\mathbf{x} = \mathbf{x}_p + c_1\mathbf{x}_1 + c_2\mathbf{x}_2$$

(*Hint:* look at the free variables in $\mathbf{x}$, $\mathbf{x}_p$, $\mathbf{x}_1$, and $\mathbf{x}_2$). Then check your answer by calculating $A\mathbf{x} - \mathbf{b}$ with MATLAB.

Optional Extra-Credit Question: Computer Graphics

Read Section 5.1 of the text and work the suggested exercises for this Section. At the MATLAB prompt type

```
H = house; plot2d(H), hold on
```

A graphics window should open and display a crude drawing of a house. The matrix `H` contains the coordinates of the endpoints of the line segments making up the drawing.

(a) Rotations: Generate a matrix Q by

```
t = pi/6; Q = [cos(t), -sin(t); sin(t), cos(t)]
```

Let Q act on the house by `plot2d(Q*H)` . How has the house been changed? Calculate $\det(Q)$. What does this tell you about the area inside the transformed house? (see Example 5 on page 222). Repeat this with `t = - pi/3` (use ↑ to save typing) and describe the result. Print the result with the three house images on the same figure.

(b) Dilations: Clear the graphics window and generate a new plot of the house as above (use ↑ to save typing). Generate a matrix D by

```
r = .9; D = [r, 0; 0, 1/r ]
```

Let D act on the house by `plot2d(D*H)`. How has the house been changed? Calculate $\det(D)$. What does this tell you about the area inside the transformed house? Repeat this with `r = .8` and describe the result. Print the result with the three house images on the same figure.

(c) Shearing Transformations: Clear the graphics window and generate a new plot of the house as above. Generate a matrix T by

```
t = 1/2; T = [1, t; 0, 1]
```

Now let T act on the house by `plot2d(T*H)`. How has the house been changed? Calculate $\det(T)$. What does this tell you about the area inside the transformed house? Repeat this with `t = - 1/2` and describe the result. What is the relation between these two transformations? Print the result with the three house images on the same figure. Print the result with the three house images on the same figure.

Final Editing of Lab Write-up: After you have worked through all the parts of the lab assignment, you will need to edit your diary file. Remove all errors and other material that is not directly related to the questions. Your write-up should only contain the required MATLAB calculations and the answers to the questions. Preview the document before printing and remove unnecessary page breaks and blank space.