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LAB 3: Determinants

In this lab you will use MATLAB to study the key aspects of the determinant of a square matrix:

- how it changes under row operations and matrix multiplication
- how it can be calculated efficiently by reduction to row echelon form
- how it can be used to obtain a formula for the inverse matrix.

Preliminaries

Reading from Textbook: Before beginning the Lab, read through Sections 3.1 and 3.2 of the text and work the suggested problems for these sections.

Tcodes: In this course you will use some instructional MATLAB *m-files* called *Tcodes*. To obtain any of these files, use a web browser (such as Netscape) and go to the Math Department Home page *http://www.math.rutgers.edu*. Click on *course materials*, then on *Math 250 Introduction to Linear Algebra* , and then on *MATLAB Teaching Codes*. You will see a directory of the m-files. Click on the particular m-files that you need. Then in the menu bar click on *Files* and *Save As*. Fill in the directory information that is requested.

For this lab you will need the Teaching Codes

```
cofactor.m,  splu.m
```

Before opening MATLAB to work on the Lab questions you should copy these codes to your diskette (or hard drive if available), as described above.

Script files: Now open MATLAB. Before beginning work on the Lab questions, use the text editor in MATLAB to create the following MATLAB *function* m-files:

(a) `rvect.m`**:** Create a *function* m-file with the commands

```
function v = rvect(m)
v = fix(10*rand(m,1));
```

(note the semicolon on the end of the second line). Save this file under the name `rvect.m` (be sure that you have set the *Path* as described in Lab 2 so that MATLAB can find this m-file). Now test the file by clicking on the MATLAB window and typing

```
v = rvect(3)
```

at the MATLAB prompt. You should get a column vector $\mathbf{v} \in \mathbf{R}^3$ with entries that are (random) integers between 0 and 9. Now type

```
u = rvect(3)
```

You will get another random column vector $\mathbf{u} \in \mathbf{R}^3$. Type `v` at the prompt. You should get the *same* vector **v** as before. Note that the name **v** in the `rvect` function file is a *local variable*; you can assign any name to the output. If you have already defined a vector **v** in your work space, it is not changed when you generate **u** by `rvect`.

(b) `rmat.m`**:** Create another function m-file with the commands

```
function A = rmat(m,n)
A = fix(10*rand(m,n));
```

(note the semicolon on the end of the second line). Save this file under the name `rmat.m` Now test the file by clicking on the MATLAB window and typing

```
A = rmat(3, 5)
```

at the MATLAB prompt. You should get a 3×5 matrix A with entries that are (random) integers between 0 and 9.

Lab Write-up: After writing and testing the script file, open your diary file (see Lab 1 for details). Begin the diary file with the comment line

```
% Math 250 MATLAB Lab Assignment #3
```

Type `format compact` so that your diary file will not have unnecessary spaces. Follow the directions as for Lab 2 to label your answers. Be sure to answer all the questions in the lab assignment. Insert comments in your diary file as you work through the assignment.

Random Seed: When you start your MATLAB session, initialize the random number generator by typing

```
rand('seed', abcd)
```

where *abcd* are the last four digits of your Student ID number. This will ensure that you generate your own particular random vectors and matrices.

BE SURE TO INCLUDE THIS LINE IN YOUR LAB WRITE-UP

The lab report that you hand in must be your own work. The following problems all use randomly generated matrices and vectors, so the matrices and vectors in your lab report will not be the same as those of other students doing the lab. Sharing of lab report files is not allowed in this course.

Question 1. The Determinant Function

(a) Row Operations: Generate a 5×5 random integer matrix `A = rmat(5,5)`. Then swap the first and second row of A to get the matrix B using the following commands:

```
B = A; B(2,:) = A(1,:); B(1,:) = A(2,:)
```

What is the relation between $\det(A)$ and $\det(B)$? Explain by general properties of the determinant function, and then check by calculating `det(A)` and `det(B)` by MATLAB.

Now let C be the matrix obtained from A by multiplying the first row of A by 10 and adding to the second row of A using the following commands:

```
C = A; C(2,:) = A(2,:) + 10*A(1,:)
```

What is the relation between $\det(A)$ and $\det(C)$? Explain by general properties of the determinant function, and then check by MATLAB.

Finally, let D be the matrix obtained from A by multiplying the first row of A by 10:

```
D = A; D(1,:) = 10*A(1,:)
```

What is the relation between $\det(A)$, $\det(D)$, and $\det(10 * A)$? Explain using general properties of the determinant function, and then check by MATLAB.

(b) Triangular Matrices: Generate a random 5×5 integer matrix `B = rmat(5,5)`. Then calculate the product

```
B(1,1)*B(2,2)*B(3,3)*B(4,4)*B(5,5)
```

This product is one term in the general formula for det(B). Can you find det(B) from this single term? Check by calculating det(B) with MATLAB. How many arithmetic operations (multiplications and additions) would be needed to calculate det(B) from formula (2) on page 141 of the text? (In counting these operations, assume that numbers are multiplied and added in pairs, so $3*4*5$ requires two multiplications, and $3+4+5+6$ requires three additions. Multiplying by $\pm$ is free.) *Note:* MATLAB does not use this formula to calculate determinants!

Now form an upper triangular matrix U by

```
U = triu(B)
```

Calculate the product `U(1,1)*U(2,2)*U(3,3)*U(4,4)*U(5,5)`. Can you find det(U) from this single term? Explain, and then check by calculating det(U) with MATLAB.

(c) Multiplicative Property: Generate a random 5×5 integer matrix `A = rmat(5,5)`. Then set `A(1,1)=0; A(2,1) = 0`. The reduction of A to row echelon form can be expressed in terms of a matrix factorization as $PA = LU$. Here P is a *permutation matrix* that expresses the row interchanges that are needed for reduction to row echelon form, and L and U are the LU factorization of PA. For the modified matrix A, why do you know that P will not be the identity matrix? You can calculate this factorization by

```
[P, L, U, sign] = splu(A)
```

(this uses the Teaching Code `splu.m`). Here `sign` gives det(P), which is the sign of the corresponding permutation.

1. Find the permutation of $\{1, 2, 3, 4, 5\}$ that when applied to the rows of the identity matrix gives P. Calculate whether it is even or odd by counting the number of inversions (see page 141 of the text). Compare your answer with the value of `sign` that MATLAB has calculated.
2. Find det(L) (answer without MATLAB, and then check by MATLAB).
3. Calculate det(U).
4. Check (by MATLAB) that $PA = LU$. Explain how this allows you to calculate det(A) from the answers to 1., 2., and 3. Check by using MATLAB.

Question 2. Cofactor Matrix and Cramer's Rule

(a) Cofactor Expansion for Determinants: The Teaching Code m-file `cofactor.m` calculates the matrix of cofactors of a square matrix. Generate a random 4×4 integer matrix `a = rmat(4,4)`. Then use MATLAB to calculate the cofactor matrix `A = cofactor(a)`. Use MATLAB to calculate the two sums

a(1,1)A(1,1) + a(1,2)A(1,2) + a(1,3)A(1,3) + a(1,4)A(1,4)

a(2,1)A(2,1) + a(2,2)A(2,2) + a(2,3)A(2,3) + a(2,4)A(2,4)

You can do this efficiently using the colon operator to work with entire rows of a and A. Use Theorem 3.9 (page 153) to explain the answers you get. Then check by using MATLAB to calculate det(a).

(b) Adjoint Matrix: Use MATLAB to calculate the sum

a(1,1)A(2,1) + a(1,2)A(2,2) + a(1,3)A(2,3) + a(1,4)A(2,4)

(the dot product of the *first* row of a with the *second* row of A'). Use Theorem 3.10 (page 155) to explain the answer you get.

The transposed cofactor matrix A' is called the *adjoint matrix* adj(a). Use MATLAB to calculate $a * A'$. Use Theorem 3.11 (page 157) to explain the answer you get. Explain how this matrix product result includes the previous calculations as a special cases.

(c) Inverse Matrix: Set `M = A'/det(a)` and calculate `a*M`. Use Corollary 3.3 (page 158) to explain the answer you get.

(d) Cramer's Rule: Generate a 3×3 random integer matrix `B = rmat(3,3)` and a random column vector **c** `= rvect(3)` Define matrices $BC1, BC2, BC3$ by replacing the first (respectively second or third) column of B by the vector **c**:

```
BC1 = B; BC2 = B; BC3 = B;
BC1(:,1) = c, BC2(:,2) = c, BC3(:,3) = c
```

Now define the column vector

```
x = [det(BC1)/det(B); det(BC2)/det(B); det(BC3)/det(B)]
```

and calculate $B * \mathbf{x}$. Explain the result using *Cramer's Rule* (Theorem 3.13, page 160).

Optional Extra-Credit Question: Polynomial Interpolation

Type `clear` before beginning this question.

(a) Vandermonde Matrices: Generate a random column vector `x = rvect(3)`. Repeat this if necessary until you obtain an **x** with three *different* integer entries. Then create the associated 3×3 *Vandermonde* matrix (see the text, page 163):

```
A = ones(3); A(:,1)= x.^2; A(:,2) = x; A
```

Note carefully the period before the exponent: `x.^2` is the vector whose entries are the *squares* of the entries of x.

The determinant of A is the product of the differences of all pairs of entries in **x**:

$$\det(A) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$$

(This is a wonderful special property of the Vandermonde matrix and it is true for general $n \times n$ Vandermonde matrices. There are very few matrices other than triangular matrices for which the determinant can be calculated by a single product.) Check this by MATLAB. Since you have taken an **x** with all entries distinct, you now know that A is invertible.

(b) Interpolating Polynomial: Read about Polynomial Interpolation in Section 1.5, page 59 of the text. Take the random vector **x** from **(a)** and generate another random vector `y = rvect(3)` (it doesn't matter whether the entries of **y** are all different or not). Use MATLAB to plot the three points $(x(1), y(1))$, $(x(2), y(2))$, and $(x(3), y(3))$ by the commands

```
axis([0,10,0,10]), hold
plot(x, y, '*')
```

There is a unique quadratic polynomial

$$p(x) = c_1 x^2 + c_2 x + c_3$$

that passes through these three points. The coefficients are the components of a vector **c** that satisfies $A\mathbf{c} = \mathbf{y}$, where A is the Vandermonde matrix from **(a)**. Calculate `c = A\y` and check that `A*c = y`. Then plot the interpolating polynomial $y = p(x)$ by

```
s = 0:.25:10; t = polyval(c,s); plot(s, t)
```

The parabola that you get should pass through all three points (marked by * on the graph). Hand in your graph and the MATLAB calculations.

Final Editing of Lab Write-up: After you have worked through all the parts of the lab assignment, you will need to edit your diary file. Remove all errors and other material that is not directly related to the questions. Your write-up should only contain the required MATLAB calculations and the answers to the questions. Preview the document before printing and remove unnecessary page breaks and blank space.