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LAB 6: Symmetric and Positive-Definite Matrices, Singular Value Decomposition, and Linear Transformations

In this lab you will use MATLAB to transform a matrix into a diagonal matrix by finding its eigenvalues and eigenvectors. Special properties of the matrix (symmetric or positive-definite) imply that the eigenvalues and eigenvectors also have special properties. You next explore the singular value decomposition of a matrix, which expresses the action of a matrix as a rotation, followed by a stretching in each coordinate direction, followed by another rotation. Finally, you will see the geometric meaning of matrices as linear transformations.

Reading from Textbook: Before beginning the Lab, read through Sections 6.4, 6.5, 6.7, and 7.1 of the text and work the suggested problems for each section.

Tcodes and Script Files: For this lab you will need the Teaching Codes

```
plot2d.m   house.m
```

Before beginning work on the Lab questions you should copy these codes from the Teaching Codes directory on the Math Department/Course Materials/Linear Algebra 250 web page to your diskette (see Lab 2 for details). You will also need the teaching code `slu.m` and the m-files `rmat.m` from Lab 3 and `rvect.m` from Lab 4. These files should already be on your diskette from previous Labs.

Lab Write-up: You should open a diary file at the beginning of each MATLAB session (see Lab 1 for details). Begin the diary file with the comment line

```
% Math 250 MATLAB Lab Assignment #6
```

Type `format compact` so that your diary file will not have unnecessary spaces. Put labels to mark the beginning of your work on each part of each question. For example,

```
% Question 1 (a) ...
       ⋮
% Question 1 (b) ...
```

and so on. Insert comments in your diary file as you work through the assignment. Be sure to answer all the questions in the lab assignment. Preview the document before printing and remove mistakes, unnecessary page breaks and blank space. Put your name, section number, and student ID number on each page. (If you have difficulty doing this using your text editor, you can write this information by hand after printing the report.)

Important: An unedited diary file without comments will get a GRADE OF ZERO as a lab writeup.

Question 1. Eigenvalues and Eigenvectors for Symmetric Matrices

(a) 2×2 **Symmetric Matrix:** Generate the matrix

```
A = [1, 3; 3, 2]/4
```

in the MATLAB command window (notice that A is a symmetric matrix). Then type `eigshow(A)`. A graphics window should open. The matrix A should be shown above the graph (in MATLAB notation). Underneath the graph the statement

```
Make A*x parallel to x
```

should appear (if it does not, then click on the `eig` button to get this statement).

Move x around the circle with the cursor and search for the *special directions* of x where Ax and x lie on a straight line. When x points in one of these directions, it is an *eigenvector* of the matrix A (the word *eigen* means *special* in German). When x is an eigenvector, $Ax = \lambda x$, and λ is the corresponding *eigenvalue* of A. Since x is a unit vector, the length of Ax is $|\lambda|$. If Ax points in the same direction as x, then $\lambda > 0$, otherwise it is negative. Find the two directions x so that Ax is parallel to x. Are these two directions perpendicular? Give a rough estimate (magnitude and sign) for the eigenvalues of A based on the graph (remember that x is a unit vector).

Now use MATLAB to calculate eigenvalues and eigenvectors of A by the commands

```
[Q, E] = eig(A)
```

(the columns of Q are the normalized eigenvectors and the diagonal entries of E are the eigenvalues). Compare the calculated eigenvalues with your graphical estimates for the eigenvalues.

Use MATLAB to verify that Q is an orthogonal matrix. What does this tell you about the angle between the two eigenvectors? Verify that $A = Q * E * Q^{\mathrm{T}}$ (see Strang, page 286, equation 6H). Close the eigshow window.

(b) 3×3 **Symmetric Matrix:** Generate a random 3×3 symmetric matrix A by

```
B = fix(5*rand(3)); A = B + B'
```

Is $A^{\mathrm{T}} = A$? Since A is a random matrix, it should have three distinct eigenvalues and corresponding eigenvectors. From the property $A^{\mathrm{T}} = A$ what can you predict about the angles between the eigenvectors?

Have MATLAB find the eigenvalues and eigenvectors of A by

```
[Q, E] = eig(A)
```

(the columns of Q are the normalized eigenvectors and the diagonal entries of E are the eigenvalues). Use MATLAB to verify that Q is an orthogonal matrix. How does this confirm your predictions about the angles between the eigenvectors? Verify that $A = Q * E * Q^{\mathrm{T}}$ (see Strang, page 286, equation 6H).

Use MATLAB to define

```
e1 = E(1,1), e2 = E(2,2), e3 = E(3,3)
```

These are the three eigenvalues of A. Now make A a symbolic matrix and calculate its characteristic polynomial $p(s)$ by

```
syms s; A = sym(A); I = eye(3); p = det(A - s*I)
```

Calculate `det(A)` by MATLAB. Check that this number is the constant term in the polynomial $p(s)$. Since the roots of $p(s)$ are e_1, e_2, e_3 and the leading term is $-s^3$, the polynomial factors as

$$(*) \qquad p(s) = -(s - e_1)(s - e_2)(s - e_3).$$

(you don't have to verify this). Hence the constant term in $p(s)$ is also `e1*e2*e3` (the product of the eigenvalues). Verify this by MATLAB (up to numerical roundoff error).

Verify that the coefficient of s^2 in $p(s)$ is `trace(A)` (the *trace* of a matrix is the sum of the diagonal entries). The factorization $(*)$ of $p(s)$ shows that this coefficient is also $e_1 + e_2 + e_3$. Verify this by MATLAB (up to numerical roundoff error).

Use $(*)$ and algebra (done by hand) to find a formula for the coefficient of s in $p(s)$ in terms of the eigenvalues e_1, e_2, e_3 (include your calculations in your write up). Then verify by MATLAB that your formula is correct for the matrix A (up to numerical roundoff error).

Question 2. Positive-Definite Matrices

(a) Tests for Positive-Definiteness: (see Strang, page 299, statement 6O) Generate a random 3×3 symmetric matrix A by

```
B = fix(5*rand(3)); A = B + B'
```

Calculate the three *upper left determinants* of A by

```
D1 = A(1,1), D2 = det(A(1:2,1:2)), D3 = det(A)
```

One test for A to be *positive definite* is

$$D_1 > 0, \quad D_2 > 0, \quad D_3 > 0$$

Does the matrix A pass this test?

Another test for A to be positive definite is that all the *eigenvalues* of A are positive. Calculate these eigenvalues by the MATLAB command `eig(A)`. Are the results of this test consistent with the first test?

A third test for A to be positive definite is that all the *pivots* of A are positive. Check this by calculating

```
[L, U] = slu(A)
```

The pivots are the diagonal entries of U (here `slu.m` is the Teaching Code that you used in Lab 2). Are the pivots all positive? Is this consistent with the previous two tests?

If the matrix A you generated failed these tests, generate more symmetric matrices until you get one that passes all the tests.

(b) Positivity of $A^{\mathrm{T}}A$: If A has independent columns, then the matrix $R = A^{\mathrm{T}}A$ is always positive definite. Check this by generating a random 4×3 matrix `A = rmat(4,3)` and setting `R = A'*A`. Calculate `eig(R)` to see that R is positive definite. Generate a random vector `x = rvect(3)` and check that `x'*R*x` is positive.

Question 3. Singular Value Decomposition

(a) Graphic Demo of SVD: Generate a random 2×2 matrix `A = rand(2,2)`. Then type `eigshow(A)` at the MATLAB prompt. A graphics window should open. Click on the `svd` button on the right side of the window. Your matrix A should appear (in MATLAB notation) in the menu bar above the graph. Underneath the graph the statement

```
Make A*x perpendicular to A*y
```

should appear (if it does not, then click on the `svd` button again). The graph shows a pair of orthogonal unit vectors **x** and **y**, together with the transformed vectors $A\mathbf{x}$ and $A\mathbf{y}$ (see Strang, page 317).

Move the pointer onto the vector **x**, and then make the pair of vectors **x**, **y** go around a circle. The transformed vectors $A\mathbf{x}$ and $A\mathbf{y}$ then move around an ellipse, but generally $A\mathbf{x}$ is not perpendicular to $A\mathbf{y}$. Search for the position of **x** and **y** so that $A\mathbf{x}$ is perpendicular to $A\mathbf{y}$. When this happens, then the *singular values* σ_1 and σ_2 of A are the lengths of the vectors $A\mathbf{x}$ and $A\mathbf{y}$. Give a rough estimate of these lengths from the graph (remember that **x** and **y** have length one).

(b) Calculation of SVD: Let A be the random matrix you generated in part (a). Use MATLAB to calculate the singular value decomposition of A by

```
[U, S, V] = svd(A)
```

Verify that $A = U * S * V'$ (up to numerical roundoff error).

The diagonal entries of S are the *singular values* σ_1, σ_2 of A. Compare the calculated singular values with your graphical estimates for σ_1, σ_2 from part (a).

The singular values are the square roots of the eigenvalues of $A^T A$. Check this by calculating

```
sqrt(eig(A'*A))
```

(c) Geometric Meaning of SVD: At the MATLAB prompt type

```
H = house; plot2d(H)
```

A graphics window should open and display a crude drawing of a house (see Figure 7.1 on page 324 of Strang; the matrix `H` contains the coordinates of the dots in the figure).

Generate an orthogonal matrix V by

```
t = pi/6; V = [cos(t), -sin(t); sin(t), cos(t)]
```

Let V^T act on the house by `plot2d(V'*H)` (be sure to use the transpose matrix `V'`). How has the house been changed?

Next generate a diagonal matrix S by

```
S = [5/4, 0; 0, 3/4 ]
```

Let S act on the rotated house by `plot2d(S*V'*H)` . How does this change the previous picture?

Now generate another orthogonal matrix U by

```
t = pi/4; U = [cos(t), -sin(t); sin(t), cos(t)]
```

Let U act on the rotated and stretched house by `plot2d(U*S*V'*H)` . How does this change the previous picture? Print this figure and include it in your lab write up.

Set `A = U*S*V'`. Use MATLAB to calculate the singular value decomposition of A by

```
[U1, S1, V1] = svd(A)
```

Verify that `U = U1, S = S1, V = V1` (up to numerical roundoff error).

Question 4. Linear Transformations

(a) Rotations: Generate a matrix Q by

```
t = pi/3; Q = [cos(t), -sin(t); sin(t), cos(t)]
```

Let Q act on the house by `plot2d(Q*H)`. How has the house been changed? Repeat this with `t = - pi/3` and describe the result. What is the relation between these two transformations?

(b) Dilations: Generate a matrix D by

```
r = 3/4; D = [r, 0; 0, 1/r ]
```

Let D act on the house by `plot2d(D*H)`. How has the house been changed? Repeat this with `r = 4/3` and describe the result. What is the relation between these two transformations?

(c) Shearing Transformations: Generate a matrix T by

```
t = 1/4; T = [1, t; 0, 1]
```

Now let T act on the house by `plot2d(T*H)`. How has the house been changed? Repeat this with `t = -1/4` and describe the result. What is the relation between these two transformations? In your lab writeup include a printed copy of the transformed house figure generated by `plot2d(T*H)` .