Revised 5/23/00

LAB 5: $A = QR$ Factorization, Determinants, and Eigenvalues/Eigenvectors

In this lab you will use MATLAB to study three topics:

1) How to transform a given set of basis vectors into an orthonormal basis (the *Gram-Schmidt Algorithm*). When the given vectors are columns of the matrix A, you obtain the $A = QR$ matrix factorization.

2) The determinant function: how it is calculated from the $PA = LU$ matrix factorization, and how to obtain A^{-1} using determinants (cofactors, Cramer's Rule)

3) Eigenvalues and eigenvectors: the characteristic polynomial of a square matrix, the roots of this polynomial (the eigenvalues), and the eigenvectors.

Reading from Textbook: Before beginning the Lab, read through Sections 4.4, 5.1, 5.2, 5.3, 6.1 and 6.2 of the text and work the suggested problems for each section.

Tcodes and Script Files: For this lab you will need the Teaching Codes

```
cofactor.m,  splu.m
```

Before beginning work on the Lab questions you should copy these codes from the Teaching Codes directory on the Math Department/Course Materials/Linear Algebra 250 web page to your diskette (see Lab 2 for more details). You will also need the m-files `rmat.m` from Lab 3 and `rvect.m` from Lab 4. These files should already be on your diskette.

Lab Write-up: You should open a diary file at the beginning of each MATLAB session (see Lab 1 for details). Begin the diary file with the comment line

```
% Math 250 MATLAB Lab Assignment #5
```

Type `format compact` so that your diary file will not have unnecessary spaces. Put labels to mark the beginning of your work on each part of each question. For example,

```
% Question 1 (a) ...
        ⋮
% Question 1 (b) ...
```

and so on.

Be sure to answer all the questions in the lab assignment. Insert comments in your diary file as you work through the assignment. After you have worked through all the parts of a lab, you will need to edit your diary file, as you did for the previous Labs.

Remove all errors and other material that is not directly related to the questions. For example, remove all material associated with writing and testing your script files.

Preview the document before printing and remove unnecessary page breaks and blank space. Put your name, section number, and student ID number on each page. (If you have difficulty doing this using your text editor, you can write this information by hand after printing the report.)

Important: An unedited diary file without comments will get a GRADE OF ZERO as a lab write up.

Question 1. Gram-Schmidt Orthogonalization and QR Factorization

(a) Dot Products: Generate three random integer vectors in $\mathbf{R}^3$ by

```
v1 = rvect(3); v2 = rvect(3); v3 = rvect(3)
```

(here `rvect.m` is the m-file from Lab 4). Calculate the dot products of these vectors:

```
v1'*v2, v1'*v3, v2'*v3
```

Are the vectors mutually orthogonal?

(b) Gram-Schmidt Orthogonalization: Let v_1, v_2, v_3 be the three vectors from part (a). Set

```
w1 = v1, w2 = v2 - ((v2'*w1)/(w1'*w1))*w1
```

Calculate the dot product of the vectors w_1 and w_2. Are these vectors mutually orthogonal (within a negligible numerical error)? Now set

```
w3 = v3 - ((v3'*w1)/(w1'*w1))*w1 - ((v3'*w2)/(w2'*w2))*w2
```

(use the up-arrow key to edit the previous command). Calculate the dot products `w1'*w3` and `w2'*w3`. Are the vectors w_1, w_2, w_3 mutually orthogonal (within a negligible numerical error)? Now rescale the vectors w_1, w_2, w_3 to have length one by using the MATLAB function `norm`:

```
q1 = w1/norm(w1), q2 = w2/norm(w2), q3 = w3/norm(w3)
```

Check (by MATLAB) that $||q_i|| = 1$ for $i = 1, 2, 3$ (here $||q||$ is the usual mathematical notation for the norm (length) of the vector q). Why are the vectors q_1, q_2, q_3 mutually orthogonal (answer without MATLAB)? Set

```
Q = [q1, q2, q3]
```

Why is $Q' * Q$ the 3×3 identity matrix? What is the *inverse matrix* Q^{-1}? Answer these questions first without MATLAB and then check with MATLAB.

(c) $A = QR$ **Factorization:** Set

```
A = [v1, v2, v3], R = Q'*A
```

Where do zeros occur in R? Verify by MATLAB that $A = Q * R$. Calculate

```
R(1,1)*q1
R(1,2)*q1 + R(2,2)*q2
R(1,3)*q1 + R(2,3)*q2 + R(3,3)*q3
```

How are these three vectors related to the vectors v_1, v_2, v_3?

Question 2. The Determinant Function

(a) Row Operations: Generate a 5×5 random integer matrix `A = rmat(5,5)` (here `rmat` is the m-file from Lab 3). Then swap the first and second row of A to get the matrix B:

```
B = A; B(2,:)  = A(1,:); B(1,:)  = A(2,:)
```

What is the relation between $\det(A)$ and $\det(B)$? Explain by general properties of the determinant function, and then check by calculating `det(A)` and `det(B)` by MATLAB.

Now let C be the matrix obtained from A by multiplying the first row of A by 10 and adding to the second row of A:

```
C = A; C(2,:)  = A(2,:)  + 10*A(1,:)
```

What is the relation between $\det(A)$ and $\det(C)$? Explain by general properties of the determinant function, and then check by MATLAB.

Finally, let D be the matrix obtained from A by multiplying the first row of A by 10:

```
D = A; D(1,:)  = 10*A(1,:)
```

What is the relation between $\det(A)$, $\det(D)$, and $\det(10 * A)$? Explain using general properties of the determinant function, and then check by MATLAB.

(b) Triangular Matrices: Generate a random 5×5 integer matrix `B = rmat(5,5)`. Then calculate the product

```
B(1,1)*B(2,2)*B(3,3)*B(4,4)*B(5,5)
```

This product is one term in the *big* formula for $\det(B)$ (see Strang, page 221, formula (7)). How many terms does the big formula have in this case? Can you find $\det(B)$ from this single term? Now form an upper triangular matrix U by

```
U = triu(B)
```

Calculate the product `U(1,1)*U(2,2)*U(3,3)*U(4,4)*U(5,5)`. How is this number related to $\det(U)$? Explain, and then check by calculating $\det(U)$ with MATLAB.

(c) Multiplicative Property: Generate a random 5×5 integer matrix `A = rmat(5,5)`. Then obtain the $PA = LU$ factorization of A by

```
[P, L, U, sign] = splu(A)
```

(this uses the Teaching Code `splu.m`).

1. What are $\det(L)$ and $\det(P)$? (answer without MATLAB, and give reasons).
2. Calculate $\det(U)$ by multiplying *five* numbers (without using the `det` function).
3. Check (by MATLAB) that $PA = LU$. Use this and the answers to 1. and 2. to find $\det(A)$. Then check by using MATLAB.

Now set `A(1,1) = 0`, calculate the $PA = LU$ factorization of this new A, and answer questions 1, 2, 3 again in this case. Explain the difference from the previous result

Question 3. Cofactor Formula for Inverse Matrix and Cramer's Rule

(a) Cofactor expansion of $\det(A)$**:** The determinant of a matrix A is a *linear function* of the first row of the matrix (when the other rows are fixed). To see this for 3×3 matrices, create a symbolic first row u by

```
syms a11 a12  a13; u = [a11  a12  a13 ]
```

(Review the end of Lab 1 for the use of symbolic variables in MATLAB.) The following commands create a 3×3 random integer matrix A and make it have symbolic variables; then the first row of A is replaced by the symbolic row u:

```
A = rmat(3,3); A = sym(A); A(1,:)  = u
```

Use MATLAB to calculate $\det(A)$. Notice that it is a linear function of the variables a_{11}, a_{12}, and a_{13}. The coefficient of each a_{ij} in this linear function is the corresponding cofactor C_{ij}. Do a hand calculation to verify this: write down the three appropriate 2×2 minors by hand and calculate their determinants with $\pm$ signs as needed (*include this hand work in your lab write up*). Then use the Teaching Code m-file `cofactor.m` to check that the coefficient of a_{1j} in $\det(A)$ is `cofactor(A,1,j)` , for $j = 1, 2, 3$.

(b) Cofactor Matrix: Let A be the 3×3 symbolic matrix that you created in part (a). Use MATLAB to calculate the cofactor matrix `C = cofactor(A)`. The *transposed cofactor matrix* C^T is then obtained by the command `C.'` (notice carefully the period between C and '. This is needed to obtain the transpose because C is a symbolic matrix). Set `I = eye(3)` and calculate `det(A)*I` by MATLAB (use the command `simplify(ans)`). Then calculate `A*C.'` and use the `simplify` command. The two expressions should be the same (see formula (7) on page 232 of Strang).

Now set `M = C.'/det(A)` and calculate $A * M$. What should the answer be? Verify your prediction by MATLAB (you will need to use the `simplify` command again).

(c) Cramer's Rule: Define a symbolic vector $b \in \mathbf{R}^3$ by

```
syms b1 b2 b3; b = [b1; b2; b3]
```

Let A be the matrix from part (a). Set `x = inv(A)*b` and verify by MATLAB that $A * x = b$ (you will need to use the `simplify` command). Then define a matrix $B1$ by replacing the first column of A by the vector b:

```
B1 = A; B1(:,1) = b
```

Cramer's Rule says that the first component `x(1)` of the vector x is the ratio `det(B1)/det(A)` (and a similar formula holds for the other components; see Strang, page 230, formula 5B). Verify this by MATLAB (you will need to use the `simplify` command). Note that `x(1)` is a *linear* function of the variables `b1, b2, b3`, but it is not a linear function of the variables `a11, a12, a13` (these appear in both the numerator and the denominator).

Question 4. Eigenvalues and Eigenvectors

(a) Graphic Demo: Type `eigshow` at the MATLAB prompt. A graphics window should open. Above the graph a matrix A is shown (in MATLAB notation). In the graph, a unit vector x and the transformed vector A are shown. Move the pointer onto the vector x, and then make x go around a circle. The transformed vector Ax then moves around an ellipse. Search for the *special directions* of x where Ax and x lie on a straight line. When x points in one of these directions, it is an *eigenvector* of the matrix A (the word *eigen* means *special* in German). When x is an eigenvector, $Ax = \lambda x$, and λ is the corresponding *eigenvalue* of A. Since x is a unit vector, the length of Ax is $|\lambda|$. If Ax points in the same direction as x, then $\lambda > 0$, otherwise it is negative.

After experimenting with the matrix that is supplied by MATLAB automatically, click on pull-down matrix selection bar and select `[0 1; 1 0]`. Move x around the circle with the cursor. How many directions do you find where Ax is parallel to x? Describe the path that Ax follows. What are the eigenvalues of A? (answer these questions by looking at the graph–no calculation needed).

Next, click on pull-down matrix selection bar and select `[0 1; -1 0]`. Move x around the circle with the cursor. Are there any directions where Ax is parallel to x? Describe the relation between x and Ax as you move x around the circle. Does A have any *real* eigenvalues? (answer these questions by looking at the graph).

Finally, click on the pull-down matrix selection bar and select `randn(2,2)` to get a random 2×2 matrix. Move x around the circle with the cursor. How many directions do you find where Ax is parallel to x? Describe the path that Ax follows. Now close the graphics window (you do not have to print out the graphs).

(b) Characteristic Polynomial: Generate a random 2×2 integer matrix `A = rmat(2,2)`. The eigenvalues of A are the roots of the *characteristic polynomial* of A. Make A a symbolic matrix and calculate its characteristic polynomial $p(s)$ by

```
syms s; A = sym(A); I = eye(2); p = det(A - s*I)
```

Verify that the constant term in the polynomial $p(s)$ is $\det(A)$. Verify that the coefficient of s is `-trace(A)` (the *trace* of a matrix is the sum of the diagonal entries). The quadratic formula implies that all the roots of $p(s)$ will be *real* if

$$(\text{trace}(A))^2 \geq 4 * \det(A).$$

Is this condition satisfied for your matrix A? Is it satisfied when

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

(this is one of the matrices you used in the eigenshow demo in part (a))?

Now generate a random 3×3 integer matrix `A = rmat(3,3)`. Make A a symbolic matrix and calculate its characteristic polynomial $p(s)$ by

```
syms s; A = sym(A); I = eye(3); p = det(A - s*I)
```

Verify that the constant term in the polynomial $p(s)$ is $\det(A)$. Verify that the coefficient of s^2 is `trace(A)` (the coefficient of s is another function of the entries of A)

(c) Eigenvalues and Eigenvectors: Generate a new random 2×2 integer matrix `A = rmat(2,2)`. Use the MATLAB command

```
[S  E] = eig(A)
```

to generate a matrix S and a diagonal matrix E. Define

```
v1 = S(:,1), v2 = S(:,2), e1 = E(1,1), e2 = E(2,2)
```

(the two columns of S and the diagonal entries of E). Then calculate `A*v1 - e1*v1` and `A*v2 - e2*v2`. What does this tell you about the vectors $v1$ and $v2$? (Treat numbers less than 10^{-10} as zero.)

Verify that `A = S*E*inv(S)` (see Strang, page 258, 6D).

Optional Extra-Credit Question

(a) Real Eigenvalues: In Question 4c, the eigenvalues of the 2 random matrix A are always real (even though A is random). Do a hand calculation (with some algebra) to show that if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is any 2×2 matrix with $bc \geq 0$, then

$$(\text{trace}(A))^2 \geq 4 \det(A)$$

Hence the roots of the characteristic polynomial of A are real (see the result in Question 4(b)).

(b) Stability: Generate a random 2×2 matrix A with nonnegative integer entries by `A = rmat(2,2)`. By part (a) you know that the eigenvalues of A are real. Calculate

```
det(A), trace(A)
```

Use this information to decide if both eigenvalues are positive, or if one is positive and the other is negative (remember that trace(A) is the sum of the eigenvalues, and det(A) is the product of the eigenvalues). Then check by the command `eig(A)` , which calculates the eigenvalues of A. Repeat until you have obtained both types of matrices: both eigenvalues positive, or one positive and one negative. (In your lab write up, only include one of each of these types of matrices). See Strang, pages 275-276 and Figure 6.3 for applications to differential equations.