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LAB 3: Complete Solution to $A\mathbf{x} = \mathbf{b}$

In this lab you will use MATLAB to study the complete solution of the linear equation $A\mathbf{x} = \mathbf{b}$, where A is a given $m \times n$ matrix, $\mathbf{b}$ is a given $m \times 1$ column vector and $\mathbf{x}$ is an unknown $n \times 1$ column vector.

Reading from Textbook: Before beginning the Lab, read through Sections 3.1, 3.2, 3.3, and 3.4 of the text and work the suggested problems for each section.

Tcodes: For this lab you will need the Teaching Codes

```
nulbasis.m, elim.m, partic.m
```

Before beginning work on the Lab questions you should copy these codes from the Tcodes directory on the Math Department Course Materials web page to your diskette (see Lab 2 form more details).

Script files: Before beginning work on the Lab questions, use the text editor in MATLAB to create the following MATLAB *function* m-file with the commands

```
function A = rmat(m,n)
A = fix(10*rand(m,n));
```

(note the semicolon on the end of the second line). Save this file under the name `rmat.m` (be sure that you have set the *Path* as described in Lab 2 so that MATLAB can find this m-file). Now test the file by clicking on the MATLAB window and typing

```
A = rmat(3, 5)
```

at the MATLAB prompt. You should get a 3×5 matrix A with entries that are (random) integers between 0 and 9. You can toggle between the MATLAB window and Editor/Debugger window to modify your script if necessary. You might find it convenient to resize the two windows so that they are displayed side by side. Use the up-arrow key $\uparrow$ to execute the command `A = rmat(3, 5)` again. You should get a new random 3×5 matrix A. Now type

```
B = rmat(4, 6)
```

You will get a random 4×6 matrix called B. Type `A` at the prompt. You should get the *same* 3×5 random matrix as before. Note that the name A in the `rmat` function file is a *local variable*; you can assign any name to the output. If you have already defined a matrix A in your work space, it is not changed when you generate B by `rmat`.

Lab Write-up: You should open a diary file at the beginning of each MATLAB session (see Lab 1 for details). Begin the diary file with the comment line

```
% Math 250 MATLAB Lab Assignment #3
```

Type `format compact` so that your diary file will not have unnecessary spaces. Put labels to mark the beginning of your work on each part of each question. For example,

```
% Question 1 (a) ...
      ⋮
% Question 1 (b) ...
```

and so on.

Be sure to answer all the questions in the lab assignment. Insert comments in your diary file as you work through the assignment. After you have worked through all the parts of a lab, you will need to edit your diary file, as you did for the previous Labs.

Remove all errors and other material that is not directly related to the questions. For example, remove the material associated with writing and testing your script files.

Preview the document before printing and remove unnecessary page breaks and blank space. Put your name, section number, and student ID number on each page. (If you have difficulty doing this using your text editor, you can write this information by hand after printing the report.)

Important: An unedited diary file without comments will get a GRADE OF ZERO as a lab writeup.

Question 1. Reduced Row Echelon Form

Before starting work on this question, type `rrefmovie` at the MATLAB prompt. You will see a step-by-step example of the row operations that transform a matrix `A` into `rref(A)`. Each pivot is chosen to be the *largest* in its column (for numerical stability), so extra row interchanges are used. Since `rref(A)` is *uniquely* determined by A, this does not affect the final answer. (Do not include this part in your lab write-up.)

(a) Random Square Matrices: The Teaching Code `elim` calculates the reduced row echelon form matrix R for A and the invertible elimination matrix E so that $EA = R$. Type

```
A = rmat(3, 3), [E, R] = elim(A)
```

at the prompt. Then calculate $E * A - R$ and check that the result is approximately 0 (it will not be exactly zero due to round-off error). The *rank* of A is the number of pivots in R (this is the same as the number of non-zero rows in R). Use MATLAB to calculate it by the command `rank(A)`. Use the up-arrow key ↑ to repeat this calculation.

Now try

```
A = rmat(5, 5), [E, R] = elim(A)
```

(throughout this Lab you can save typing by using the up-arrow key and editing the previous commands). Then calculate $E * A - R$ and check that the result is approximately 0. Calculate `rank(A)`. Use the up-arrow key ↑ to repeat this calculation.

Based on this evidence, what do you expect to get as the reduced row echelon form R of a *completely random* $n \times n$ matrix A? What do you expect for `rank(A)`? What is the relation between the matrices E and A?

(b) Random Thin Matrices: Now we study matrices A with more rows than columns. Give the command

```
A = rmat(5, 3), [E, R] = elim(A)
```

(you can do this by editing a line from part (a)). Repeat this calculation using the up-arrow key. What are the positions of the pivots in R both times? How many nonzero rows does A have? Is this the same number as `rank(A)`? Calculate `inv(E)`. How is this matrix related to A?

Try another size of thin matrices with

```
A = rmat(7, 3), [E, R] = elim(A)
```

Repeat this calculation with the up-arrow key. What are the positions of the pivots in R both times? How many nonzero rows does A have? Is this the same number as `rank(A)`? Calculate `inv(E)`. How is this matrix related to A?

Based on this evidence, what do you expect to get as the reduced row echelon form R of a *completely random* $m \times n$ matrix A when $m > n$? What do you expect for `rank(A)`?

(c) Random Fat Matrices: Now we study matrices A with more columns than rows. Give the command

```
A = rmat(3, 5), [E, R] = elim(A)
```

Repeat this calculation using the up-arrow key. Which entries in the matrix R are the same both times? Which are different? Which columns of R contain the pivots? Now calculate

```
B = A(:,1:3), E*B
```

Describe in words how the square matrix B is formed from rectangular matrix A. What is the relation between B and the elimination matrix E? Calculate `rank(A)`. Is it the same as the number of nonzero rows of A?

The MATLAB command `rref(A)` calculates the reduced row echelon form of A. Try

```
A = rmat(3, 7), R = rref(A)
```

and repeat this calculation using the up-arrow key. Which entries in the matrix R are the same both times? Which are different? Which columns of R are the pivot columns? What is `rank(A)`?

Based on this evidence, what do you expect to get as the reduced row echelon form R of a *completely random* $m \times n$ matrix A when $m < n$. How many columns of R will have pivots? Which entries of R would you know before any computation? Which entries of R are unpredictable until after the computation? What do you expect for `rank(A)`?

(d) Partly Random Fat Matrices: Generate a matrix A and its reduced row echelon form R by

```
B = rmat(3, 2); C= rmat(3,2); A = [B, 3*B, C], R = rref(A)
```

Use the up-arrow key to repeat this calculation. Which entries in the matrix R are the same both times? Which are different? What are the positions of the pivots in R? How many columns of R have pivots?

Does R fit the pattern that you predicted in part (c) when A is a *completely random* 3×6 matrix? Explain why A is *not* a completely random matrix by showing how columns 3 and 4 of A are determined by columns 1 and 2.

Question 2. Nullspace of A

(a) Special Solutions to $A\mathbf{x} = 0$ (A random and fat): The MATLAB Teaching Code `nulbasis.m` calculates the *special solutions* to the equation $A\mathbf{x} = 0$. Try it first with

```
A = rmat(3, 5), R = rref(A), N = nulbasis(A)
```

Which columns of R correspond to the free variables? Type

```
s1 = N(:,1),   s2 = N(:,2)
```

The first column $\mathbf{s}_1$ of N is the first *special solution* to $Ax = 0$ and the second column $\mathbf{s}_2$ of N is the second special solution. Check that `A*s1 = 0 and A*s2 = 0` (Due to roundoff error, the vectors will not be exactly zero, but can have small components on the order of 10^{-14}.) Use the up arrow $\uparrow$ to generate a new A and to repeat the calculation. Which component of $\mathbf{s}_1$ is *always* 1? Which component of $\mathbf{s}_1$ is *always* 0? Which component of $\mathbf{s}_2$ is *always* 1? Which component of $\mathbf{s}_2$ is *always* 0?

From Question 1 you know that R will have the block matrix form

$$R = [I \quad F], \quad \text{where } I = 3 \times 3 \text{ identity matrix}, \quad F = 3 \times 2 \text{ matrix}$$

(since A is a *general* 3×5 matrix). Write N in block form using the matrix F (do not use MATLAB for this).

(b) Special Solutions to $A\mathbf{x} = 0$ (A partly random): Now try

```
B = rmat(3, 2); C= rmat(3,2); A = [B, 3*B, C], R = rref(A), N =
nulbasis(A)
```

(you can use the up-arrow key $\uparrow$ and edit the similar line from part (d) of Question 1). Which columns of R contain the pivots? Which columns of R correspond to the free variables? Give the command

```
s1 = N(:,1),   s2 = N(:,2),   s3 = N(:,3)
```

The first column $\mathbf{s}_1$ of N is the first *special solution* to $Ax = 0$, the second column $\mathbf{s}_2$ of N is the second special solution, and the third column $\mathbf{s}_3$ is the third special solution.

Use the up arrow $\uparrow$ to generate a new A and to repeat the calculation of N. Which component of $\mathbf{s}_1$ is *always* 1? Which components of $\mathbf{s}_1$ are *always* 0?

(c) General Solutions to $A\mathbf{x} = 0$: Take one of the matrices A and the special solutions $\mathbf{s}_1$, $\mathbf{s}_2$, and $\mathbf{s}_3$ to $A\mathbf{x} = 0$ that you generated in part (b). Generate a random linear combination of the vectors $\mathbf{s}_1$, $\mathbf{s}_2$, and $\mathbf{s}_3$ by

```
x = rand(1)*s1 + rand(1)*s2 + rand(1)*s3
```

Note that each occurrence of `rand(1)` generates a different random coefficient. Check that $A\mathbf{x} = 0$ and $R\mathbf{x} = 0$ (instead of zero you may get a small nonzero number, due to roundoff error).

Let x be a column vector of the form

$$\mathbf{x} = [x_1,\, x_2,\, 7,\, -9,\, x_5,\, 8]^T$$

($\mathbf{x}$ is written using the transpose to save space). Suppose that $\mathbf{x}$ satisfies $A\mathbf{x} = 0$. Find scalars c_1, c_2 and c_3 so that

$$\mathbf{x} = c_1\mathbf{s}_1 + c_2\mathbf{s}_2 + c_3\mathbf{s}_3$$

(this doesn't require MATLAB or even any calculation; just look at each free variable in $\mathbf{x}$, $\mathbf{s}_1$, $\mathbf{s}_2$, and $\mathbf{s}_3$). After choosing the right values for these scalars, check your answer by calculating $\mathbf{x}$ and $A\mathbf{x}$ with MATLAB.

(d) Solutions to $A\mathbf{x} = 0$ (A random and thin): When A is thin, then the equation $A\mathbf{x} = 0$ is *overdetermined* (more equations than unknowns). In general there will not be any solution other than the *trivial solution* $\mathbf{x} = 0$. Try

```
A = rmat(5, 3), R = rref(A), N = nulbasis(A)
```

Are there any columns of R corresponding to free variables? What vectors are in the null space of A?

Question 3. Solving $A\mathbf{x} = \mathbf{b}$

(a) Particular Solution (A **fat**)**:** Generate a random 3×5 matrix A and its reduced row echelon form R by

```
A = rmat(3,5),   R=rref(A)
```

Now generate a random 3×1 vector $\mathbf{b}$ and use the Teaching Code `partic` to find a particular solution to $A\mathbf{x} = \mathbf{b}$ by

```
b = rmat(3,1),   x = partic(A, b)
```

Check that $A\mathbf{x} = \mathbf{b}$. Repeat for another random vector $\mathbf{b}$, using the same matrix A. What entries in x are zero both times? Which columns of R correspond to free variables? Calculate `rank(A)` and `rank([A, b]` (the augmented matrix). Does the equation $A\mathbf{x} = \mathbf{b}$ have a solution for *every* possible $\mathbf{b}$ in this case? Why?

(b) Particular Solution (A **thin**)**:** Generate a random 5×3 matrix `A = rmat(5,3)`. The following MATLAB command will generate a random 5×1 vector $\mathbf{b}$ and try to find a particular solution to $A\mathbf{x} = \mathbf{b}$:

```
b = rmat(5,1),   x = partic(A, b)
```

What answer did you get for $\mathbf{x}$? Calculate `rank(A)` and `rank([A, b])`. Do you get the same relation between the ranks as you got in part (a)? Use this information to explain why there is no solution to $A\mathbf{x} = \mathbf{b}$ for a completely random choice of $\mathbf{b}$.

Now use the (partly) random vector

```
b = rand(1)*A(:,1) + rand(1)*A(:,2) + rand(1)*A(:,3)
```

and calculate `x = partic(A, b)` . Explain why the special form of $\mathbf{b}$ guarantees that there always is a solution $\mathbf{x}$, no matter what the random coefficients might be. Since there is a solution to $A\mathbf{x} = \mathbf{b}$ for this $\mathbf{b}$, what is the rank of the augmented matrix `[A, b]`? Answer first without calculation, and then check using MATLAB.

(c) General Solution: Generate a random 3×5 matrix `A=rmat(3,5)`, a random 3×1 vector `b= rmat(3,1)`, and `N=nulbasis(A)`. Set `s1 = N(:,1)` and `s2 = N(:,2)` as in Question 2 and calculate `xp = partic(A,b)`. Now at a new MATLAB prompt, type

```
xn = rand(1)*s1 + rand(1)*s2,   x = xp + xn,   A*x,   b
```

Use the up-arrow key to execute this line again. Notice that $A\mathbf{x} = \mathbf{b}$ both times, even though the vector $\mathbf{x}_n$ changes randomly (always remaining in the subspace $\mathbf{N}(A)$).

Now let $\mathbf{x}$ be a column vector of the form

$$\mathbf{x} = [x_1,\, x_2,\, x_3,\, -9,\, 8]^{\mathrm{T}}$$

($\mathbf{x}$ is written using the transpose to save space). Suppose that $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{b}$. Just as in part (c) of Question 2, you can find scalars c_1 and c_2 so that

$$\mathbf{x} = \mathbf{x}_p + c_1\mathbf{s}_1 + c_2\mathbf{s}_2$$

(no calculation needed; just look at the free variables in $\mathbf{x}$, $\mathbf{x}_p$, $\mathbf{s}_1$, and $\mathbf{s}_2$). Then check your answer by calculating $\mathbf{x}$ and $A\mathbf{x}$ with MATLAB.