(20) 1. Let $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 4 \end{bmatrix}$.

(a) For each of the four subspaces associated with A, find the dimension of the subspace and a basis of it.

<u>Solution :</u> *Row-reduction yields* $R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. $C(A)$ *has dimension* 2 *and a basis consists of the first two columns of A.* $R(A)$ *has dimension* 2 *and a basis consists of the first two rows of R.* $N(A)$ *has dimension* 2 *and has as a basis the special solutions, namely* $(3, -3, 1, 0)$ *and* $(4, -4, 0, 1)$. *Finally* $A^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & 3 \\ 0 & 4 & 4 \end{bmatrix}$ *row-reduces to* $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, *so* $N(A^T)$ *has dimension* 1 *and a basis consists of the special solution* $(1, -1, 1)$.

(b) Of these four subspaces, which are orthogonal complements of which?

<u>Solution :</u> $N(A)$ *is the orthogonal complement in* $\mathbf{R}^4$ *of* $R(A)$ *(more precisely, of* $C(A^T)$*), and* $N(A^T)$ *is the orthogonal complement in* $\mathbf{R}^3$ *of* $C(A)$.

(c) What is the largest number of linearly independent vectors which can be found in $\mathbf{R}^4$ which are perpendicular to all the rows of A? Explain briefly.

<u>Solution :</u> *The vectors perpendicular to all the rows of A are the vectors in* $N(A)$. *Since* $\dim(N(A)) = 2$, *the largest number is* 2.

(20) 2. Circle **T**rue or **F**alse. Explain BRIEFLY in each case to receive credit.

T F (a) Any 6 vectors in $\mathbf{R}^5$ must span $\mathbf{R}^5$.

<u>Solution :</u> *False. There are many counterexamples. The* 6 *vectors could all be* $\mathbf{0}$, *or could all be scalar multiples of one another, or could all lie in a plane, etc., etc.*

T F (b) Let V be the nullspace of the matrix $[1 \quad 2 \quad 3 \quad 4]$. Then any three vectors in V which span V must be linearly independent.

<u>Solution :</u> *True. The rank of this matrix is* 1, *so* $\dim(N(A)) = \#$ *of columns – rank* $= 3$. *In any 3-dimensional vector space, any three vectors which span the space are linearly independent.*

T F (c) Suppose that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are vectors in $\mathbf{R}^3$. If $\mathbf{v}_2$ is not a scalar multiple of $\mathbf{v}_3$, and if $\mathbf{v}_1$ is not a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$, then $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ must span $\mathbf{R}^3$.

<u>Solution :</u> *False but almost true. In the sequence* $\mathbf{v}_3, \mathbf{v}_2, \mathbf{v}_1$, *none of the vectors is a linear combination of its predecessors. As long as all three vectors are nonzero, they therefore form a linearly independent set. Since there are three of them and* $\dim(\mathbf{R}^3) = 3$, *they therefore span* $\mathbf{R}^3$.

However, it is possible that $\mathbf{v}_3 = \mathbf{0}$, *and then the three vectors cannot span* $\mathbf{R}^3$. *Thus the statement is false.*

T F (d) The value of $\det\begin{bmatrix}4 & -2 & 2 & 0\\ -2 & 1 & -1 & 7\\ 5 & 0 & 11 & a_{34}\\ 2 & 4 & 6 & 7\end{bmatrix}$ does not depend on the value of a_{34}. (Hint: expand on a row or column containing a_{34}.)

Solution : *True. Expand on the last column. All the cofactors are independent of* a_{34} *since one crosses out the last column. Furthermore,* $C_{34} = -\det\begin{bmatrix}4 & -2 & 2\\ -2 & 1 & -1\\ 2 & 4 & 6\end{bmatrix} = 0$ *since the first two rows are linearly dependent. Therefore the determinant is* $0 + 7C_{24} + a_{34}C_{34} + 7C_{44} = 7C_{24} + 7C_{44}$, *which is independent of* a_{34}.

(10) 3. Let $A = \begin{bmatrix}2 & -1 & -2 & 2\\ 3 & 0 & 1 & 2\\ 1 & 1 & 0 & 3\\ 2 & 0 & -1 & 2\end{bmatrix}$. What is $\det A$?

Solution : *Add the third row to the first, then expand on the second column:* $\det A = \det\begin{bmatrix}3 & 0 & -2 & 5\\ 3 & 0 & 1 & 2\\ 1 & 1 & 0 & 3\\ 2 & 0 & -1 & 2\end{bmatrix} = -\det\begin{bmatrix}3 & -2 & 5\\ 3 & 1 & 2\\ 2 & -1 & 2\end{bmatrix}$. *Pivot on the middle* 1, *expand on second column:* $\det A = -\det\begin{bmatrix}9 & 0 & 9\\ 3 & 1 & 2\\ 5 & 0 & 4\end{bmatrix} = -\det\begin{bmatrix}9 & 9\\ 5 & 4\end{bmatrix} = 9$.

(10) 4. Find a basis B of the vector space spanned by $\begin{bmatrix}1\\3\\5\end{bmatrix}$, $\begin{bmatrix}1\\2\\3\end{bmatrix}$ and $\begin{bmatrix}1\\1\\1\end{bmatrix}$. Then find additional vector(s) which together with B form a basis of $\mathbf{R}^3$.

Solution : *Row-reduce:* $\begin{bmatrix}1 & 1 & 1\\ 3 & 2 & 1\\ 5 & 3 & 1\end{bmatrix} \to \begin{bmatrix}1 & 1 & 1\\ 0 & -1 & -2\\ 0 & 0 & 0\end{bmatrix}$. *Since the first two columns are the pivot columns, a basis* B *of our vector space is* $\begin{bmatrix}1\\3\\5\end{bmatrix}$, $\begin{bmatrix}1\\2\\3\end{bmatrix}$. *In order to obtain a basis of* $\mathbf{R}^3$ *we need only add one vector which is not a linear combination of these two. There are lots of correct answers. One of them is* $\begin{bmatrix}1\\3\\5\end{bmatrix}$, $\begin{bmatrix}1\\2\\3\end{bmatrix}$, $\begin{bmatrix}1\\0\\0\end{bmatrix}$.

(10) 5. Find all the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix}2 & 2\\ 2 & -1\end{bmatrix}$.

Solution : $\det A - \lambda I = \det\begin{bmatrix}2-\lambda & 2\\ 2 & -1-\lambda\end{bmatrix} = (2-\lambda)(-1-\lambda) - 2\cdot 2 = \lambda^2 - \lambda - 6$. *The eigenvalues are the roots of* $\lambda^2 - \lambda - 6 = 0$, *that is,* $\lambda = 2$ *and* $\lambda = -2$.

The eigenvectors of A corresponding to $\lambda = 3$ are the nonzero solutions $\mathbf{v}$ of $(A-3I)\mathbf{v} = \mathbf{0}$. These are found by Gaussian elimination: $\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$. *The general solution is* $\mathbf{v} = \begin{bmatrix} 2y \\ y \end{bmatrix} = y\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. *The eigenvectors for $\lambda = 3$ are the vectors* $y\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $y \neq 0$. *Similarly the eigenvectors for $\lambda = -2$ arise from* $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}$ *and so are the vectors* $y\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$, $y \neq 0$.

(10) 6. Use Cramer's Rule to find the value of x in the solution of

$$\begin{aligned} 3x+2y+\ \ z &= 0 \\ 2x-\ \ y-5z &= 0 \\ -4x+5y+7z &= 1 \end{aligned}$$

<u>Solution :</u> $x = \dfrac{\det\begin{bmatrix} 0 & 2 & 1 \\ 0 & -1 & -5 \\ 1 & 5 & 7 \end{bmatrix}}{\det\begin{bmatrix} 3 & 2 & 1 \\ 2 & -1 & -5 \\ -4 & 5 & 7 \end{bmatrix}} = \dfrac{-9}{72} = -\dfrac{1}{8}.$

(10) 7. Find the projection of $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ on the subspace of $\mathbf{R}^4$ spanned by $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}$.

<u>Solution :</u> *Let* $A = \begin{bmatrix} 1 & 1 \\ -2 & 1 \\ 1 & -2 \\ 1 & 1 \end{bmatrix}$ *and* $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. *The projection is*

$$P(\mathbf{b}) = A(A^TA)^{-1}A^T\mathbf{b} = A\begin{bmatrix} 7 & -2 \\ -2 & 7 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{1}{45}A\begin{bmatrix} 7 & 2 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.8 \\ -0.4 \\ -0.4 \\ 0.8 \end{bmatrix}$$

(10) 8. Find the straight line $y = mx + b$ which is the best approximation (in the least-squares sense) to the data points $(-2,3)$, $(0,2)$, $(2,-1)$.
<u>Solution :</u> *The three data points give the inconsistent system*

$$\begin{aligned} -2m + b &= 3 \\ 0m + b &= 2 \\ 2m + b &= -1 \end{aligned}$$

or $A\begin{bmatrix} m \\ b \end{bmatrix} = \mathbf{b}$, *where* $A = \begin{bmatrix} -2 & 1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}$ *and* $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$. *The least-squares solution is*

$$\begin{bmatrix} m \\ b \end{bmatrix} = (A^T A)^{-1} A^T \mathbf{b} = \begin{bmatrix} 8 & 0 \\ 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} -2 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4/3 \end{bmatrix}.$$

Thus, the best line is $y = -x + \frac{4}{3}$.

(10) 9. Let $A = \begin{bmatrix} 2 & 1 \\ -2 & -3 \\ 1 & 1 \end{bmatrix}$. Apply the Gram-Schmidt process to the two columns of A, and then determine the matrices Q and R which make up the QR-factorization of A.

<u>Solution :</u> *From* $\mathbf{a} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ *and* $\mathbf{b} = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$, *we get* $\mathbf{A} = \mathbf{a} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ *and*

$\mathbf{B} = \mathbf{b} - \frac{\mathbf{b} \bullet \mathbf{A}}{\mathbf{A} \bullet \mathbf{A}}\mathbf{A} = \mathbf{b} - \frac{9}{9}\mathbf{A} = \mathbf{b} - \mathbf{A} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$. *The final stage of Gram-Schmidt yields*

$$\mathbf{q}_1 = \frac{1}{\|\mathbf{A}\|}\mathbf{A} = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix} \text{ and } \mathbf{q}_2 = \frac{1}{\|\mathbf{B}\|}\mathbf{B} = \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix},$$

$$\text{so } Q = \begin{bmatrix} 2/3 & -1/\sqrt{2} \\ -2/3 & -1/\sqrt{2} \\ 1/3 & 0 \end{bmatrix} \text{ and } R = Q^T A = \begin{bmatrix} 3 & 3 \\ 0 & \sqrt{2} \end{bmatrix}.$$