Illustration of methods of solving a *square* system of linear equations $A\mathbf{x} = \mathbf{b}$

We work the same example: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 4 & 5 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 7 \\ -1 \\ 13 \end{bmatrix}$

by various methods. In this example the coefficient matrix A is invertible. (In examples where A is not invertible, methods #1 and #2 may still be used, and yield no solutions or infinitely many solutions.) The LU-method requires more: it can only be used if A has an LU-factorization!

1. Gaussian (forward) elimination and back-substitution:

$$\begin{array}{r} x + y + z = 7 \\ x + 2y + 4z = -1 \\ 2x + 4y + 5z = 13 \end{array} \xrightarrow{GE} \begin{array}{r} x + y + z = 7 \\ y + 3z = -8 \\ 2y + 3z = -1 \end{array} \xrightarrow{GE} \begin{array}{r} x + y + z = 7 \\ y + 3z = -8 \\ -3z = 15 \end{array} \xrightarrow{BS} \begin{array}{r} z = -5 \\ y = -8 - 3z = 7 \\ x = 7 - y - z = 5 \end{array}$$

The same in matrix form (compact!):

$$\begin{bmatrix} 1 & 1 & 1 & 7 \\ 1 & 2 & 4 & -1 \\ 2 & 4 & 5 & 13 \end{bmatrix} \xrightarrow{GE} \begin{bmatrix} 1 & 1 & 1 & 7 \\ 0 & 1 & 3 & -8 \\ 0 & 2 & 3 & -1 \end{bmatrix} \xrightarrow{GE} \begin{bmatrix} 1 & 1 & 1 & 7 \\ 0 & 1 & 3 & -8 \\ 0 & 0 & -3 & 15 \end{bmatrix} \xrightarrow{BS} \begin{array}{r} z = -5 \\ y = -8 - 3z = 7 \\ x = 7 - y - z = 5 \end{array}$$

2. Gauss-Jordan (forward and backward) elimination, just in matrix form:

$$\begin{bmatrix} 1 & 1 & 1 & 7 \\ 1 & 2 & 4 & -1 \\ 2 & 4 & 5 & 13 \end{bmatrix} \xrightarrow{GJ} \begin{bmatrix} 1 & 1 & 1 & 7 \\ 0 & 1 & 3 & -8 \\ 0 & 2 & 3 & -1 \end{bmatrix} \xrightarrow{GJ} \begin{bmatrix} 1 & 0 & -2 & 15 \\ 0 & 1 & 3 & -8 \\ 0 & 0 & -3 & 15 \end{bmatrix} \xrightarrow{GJ} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -5 \end{bmatrix} \text{ so } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{x} = \begin{bmatrix} 5 \\ 7 \\ -5 \end{bmatrix}$$

3. If A is invertible, then $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 2 & 4 & 5 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{GJ} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 & 0 \\ 0 & 2 & 3 & -1 & 0 & 1 \end{bmatrix} \cdots \xrightarrow{GJ} \cdots \begin{bmatrix} 1 & 0 & 0 & 2 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2 & \frac{1}{3} & -\frac{2}{3} \\ -1 & -1 & 1 \\ 0 & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 7 \\ -1 \\ 13 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ -5 \end{bmatrix}$$

4. Find $A = LU$, solve $L\mathbf{c} = \mathbf{b}$ by forward substitution and then $U\mathbf{x} = \mathbf{c}$ by back substitution:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 4 & 5 \end{bmatrix} \xrightarrow{L_{21} = 1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 2 & 4 & 5 \end{bmatrix} \xrightarrow{L_{31} = 2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 3 \end{bmatrix} \xrightarrow{L_{32} = 2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -3 \end{bmatrix} = U, \text{ so } L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ 13 \end{bmatrix} \xrightarrow{FS} \mathbf{c} = \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 7 \\ -1 - d \\ 13 - 2d - 2e \end{bmatrix} = \begin{bmatrix} 7 \\ -8 \\ 15 \end{bmatrix} \quad \text{(find } d\text{, then } e\text{, then } f\text{)}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -8 \\ 15 \end{bmatrix} \xrightarrow{BS} \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 - y - z \\ -8 - 3z \\ -5 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ -5 \end{bmatrix} \quad \text{(find } z\text{, then } y\text{, then } x\text{)}$$

Comments:

a) Roughly speaking, #1 and #4 use the fewest "flops" (arithmetic operations). #2 saves the back-substitution in #1 but has twice as much pivot work to do so is more laborious than #1 or #4. Even more pivoting is involved in #3, in the calculation of A^{-1} (camouflaged by the "…" above).

b) If A^{-1} is known in advance then #3 is super-fast. So is #4 if the LU-factorization of A is known in advance (for instance from solving some other system of linear equations with the same A, some other time).

c) If A does not have an LU-factorization but you can factor $PA = LU$, then PA has an LU-factorization and you can use #4 to solve $PA\mathbf{x} = P\mathbf{b}$, which is equivalent to $A\mathbf{x} = \mathbf{b}$. That is, solve $L\mathbf{c} = P\mathbf{b}$ and then $U\mathbf{x} = \mathbf{c}$.