NOTE: The final exam will cover the entire course. You already have review problems concerning the material up to the second midterm. This sheet supplements them with problems on the material since the second midterm.

1. Find the general solution of the system of constant-coefficient first order homogeneous linear differential equations

$$\begin{aligned} y_1' &= 5y_1 - 8y_2 - 2y_3 \\ y_2' &= 6y_1 - 9y_2 - 2y_3 \\ y_3' &= -3y_1 + 4y_2 \end{aligned}$$

(Hint. One eigenvalue is -1.)

2. Find vectors in $\mathbf{R}^2$ which point in the directions of the axes of symmetry of the curve

$$8x^2 - 4xy + 5y^2 = 156.$$

Is the curve an ellipse or a hyperbola?

3. Let $A_t = \begin{bmatrix} 2 & t \\ 2 & 0 \end{bmatrix}$. Here t is a parameter which can take on any real value.

(a) Find the eigenvalues and eigenvectors of A_t (in terms of t).

(b) Find a value $t = t_0$ for which A_{t_0} is not diagonalizable. Is this value unique? Explain.

(c) As t approaches t_0, what happens to a pair of linearly independent eigenvectors of A_t?

(d) Explain why the geometric multiplicity of each eigenvalue of A_t is 1, regardless of the value of t.

4. True or false? Give a counterexample and a correction for each false statement, and give an explanation for each true statement.

(a) If A is a diagonalizable real square matrix, then the singular values of A are the same as the eigenvalues of A.

(b) If a square matrix has all positive eigenvalues, then its diagonal entries must all be positive.

(c) If the eigenvalues of a 5×5 matrix A are 1, 2, 3, and 4, and the eigenvalue 4 occurs with algebraic multiplicity at least 2, then A is diagonalizable if and only if $A - 4I$ has rank 3.

(d) Every real matrix has a singular value decomposition.

(e) If a square matrix A is not invertible, then A is not diagonalizable.

5. Find a SVD of $\begin{bmatrix} \sqrt{2} & 1 \\ 0 & \sqrt{2} \end{bmatrix}$. (Give U, Σ and V.)

6. Find $\lim_{n\to\infty} \begin{bmatrix} 0.3 & 0.4 \\ 0.7 & 0.6 \end{bmatrix}^n$.

7. Diagonalize $\begin{bmatrix} -1 & 6 \\ -3 & 5 \end{bmatrix}$.