1. For each of the four subspaces associated with $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & 0 & 1 \\ 2 & 7 & 12 \end{bmatrix}$, find a basis and the dimension.

2. Let $A = \begin{bmatrix} 3 & 2 & 6 & 1 & 0 \\ 2 & 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 3 & 0 \\ 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}$.

(a) Using appropriate row/column operations and row/column expansions, find $\det(A)$ in five minutes or less.

(b) How do you know that A is invertible?

(c) Find five entries of A^{-1} as quickly as possible.

3. What is the effect on the determinant of a three by three matrix A

(a) if all entries are divided by 2;

(b) if the last row is made the top row (moving the other two rows down);

(c) if the first row is replaced by three times itself minus the second row;

(d) if the first row is replaced by itself minus three times the second row?

4. Which of the following sets span $\mathbf{R}^n$ for the given value of n? Which are linearly independent?

(a) $(n = 3)$ $\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$

(b) $(n = 5)$ $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} \right\}$

(c) $(n = 4)$ $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

5. From among the vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$ choose a basis of the vector space which they span. What is the dimension of this vector space? Then find additional vectors which together with this basis form a basis of $\mathbf{R}^4$.

6. True or false? Correct the false statements.

(a) If 100 vectors in $\mathbf{R}^{100}$ are linearly dependent, then they do not span $\mathbf{R}^{100}$.

(b) If 100 vectors in $\mathbf{R}^{100}$ do not span $\mathbf{R}^{100}$, then one of them (at least) is a linear combination of the others.

(c) Given three vectors in $\mathbf{R}^3$, no two of which are scalar multiples of each other, the three vectors must form a basis of $\mathbf{R}^3$.

(d) If 100 vectors in $\mathbf{R}^{100}$ are all orthogonal to $(1, 1, \ldots, 1)$, then one of them (at least) is a linear combination of the others.

(e) If A is a 3×4 matrix whose rows are linearly independent, and B is a 4×2 matrix such that $AB = 0$, then the columns of B must be linearly dependent.

7. Complete the following sentence for each of the four subspaces associated with a 100×200 matrix A of rank 98:

The subspace ________ has dimension ________ and consists of vectors with ________ entries.

8. Suppose that V is a vector space of column vectors and that two bases of V are given: $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and $\mathbf{w}_1, \ldots, \mathbf{w}_m$. Let A be the matrix $[\mathbf{v}_1 \ldots \mathbf{v}_n \mathbf{w}_1 \ldots \mathbf{w}_m]$.

(a) What can you say (if anything) about m and n?

(b) What can you say (if anything) about the result of applying Gauss-Jordan elimination to A?

(c) What can you say (if anything) about the result of applying Gauss-Jordan elimination to A^T?

9. Use Cramer's Rule if possible to find the value of z in the solution of the following system. At what point in your calculation do you learn whether Cramer's Rule is applicable?

$$\begin{aligned} 3x + 2y + 3z &= 0 \\ 2x - 3y - 5z &= 0 \\ -4x + 5y + 7z &= 1 \end{aligned}$$

10. Find the straight line $y = mx + b$ which is the least-squares best fit to the four data points $(-2, 0)$, $(-1, 2)$, $(1, 5)$, $(2, 8)$.

11. Find the projection of $(0, 0, 0, 1)$ on

(a) the subspace of $\mathbf{R}^4$ spanned by $(2, 0, 1, -1)$

(b) the subspace of $\mathbf{R}^4$ spanned by $(1, 1, 1, 1)$ and $(2, 0, 1, -1)$.

12. Apply the Gram-Schmidt procedure to the vectors $(2, 2, 2, 2), (0, 2, 2, 0), (2, 0, 0, 0)$. Then find the QR-factorization of $\begin{bmatrix} 2 & 0 & 2 \\ 2 & 2 & 0 \\ 2 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$.

13. Find all of the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix}$.