

1 Definition A vector space of column vectors is a non-empty set V of column vectors such that any linear combination of vectors in V is again a vector in V .

2 Examples For any n , \mathbf{R}^n is a vector space. For any matrix A , $N(A)$ is a vector space. For any matrix A , $C(A)$ is a vector space.

3 Definition A vector space is a non-empty set V of mathematical objects for which addition and scalar multiplication are defined and satisfy the “eight reasonable rules”, and such that any linear combination of objects in V is again an object in V . The objects in a vector space are usually referred to as “vectors” even if they aren’t literally row or column vectors.

4 Examples For any matrix A , $R(A)$ is a vector space of row vectors. The set $M_{2,2}$ of all 2×2 matrices is a vector space. For given m and n , the set $M_{m,n}$ of all $m \times n$ matrices is a vector space. The set $Sym_{2,2}$ of all 2×2 symmetric matrices is a vector space. The set of all polynomials of degree at most 2 is a vector space. For any homogeneous linear ordinary differential equation with constant coefficients, such as $y'''' - y = 0$, the set of all functions which are solutions is a vector space. (The same holds without the constant coefficient assumption but one must be careful about specifying the domains of the functions.)

5 Definition If V is a vector space and $\mathbf{v}_1, \dots, \mathbf{v}_n$ are vectors, then $\mathbf{v}_1, \dots, \mathbf{v}_n$ **span** V if and only if

- a) $\mathbf{v}_i \in V$ for each $i = 1, \dots, n$; **and**
- b) Every vector in V is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_n$.

6 Examples

$(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ span \mathbf{R}^3 .

$(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(2, 3, 4)$ span \mathbf{R}^3 .

$(2, 3, 1)$, $(0, 3, 0)$ and $(6, -1, 3)$ do not span \mathbf{R}^3 (in any linear combination of them, the first entry is twice the third, so $(1, 0, 0)$, for example, is not a linear combination of them).

$(1, 2, 3)$ and $(3, 5, 7)$ do not span \mathbf{R}^3 ($(0, 0, 1)$ cannot be expressed as a linear combination of them).

The vector space $M_{2,2}$ is spanned by the four matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

The vector space $Sym_{2,2}$ is spanned by the three matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

For a given matrix A , the special solutions of $A\mathbf{x} = 0$ span $N(A)$.

For a given matrix A , the pivot columns of \mathbf{A} span $C(A)$.

For a given matrix A , the nonzero rows of its rref R span $R(A)$.

7 Theorem If $\mathbf{v}_1, \dots, \mathbf{v}_n$ are column vectors in \mathbf{R}^m , then they span \mathbf{R}^m if and only if the $m \times n$ matrix $[\mathbf{v}_1 \cdots \mathbf{v}_n]$ has rank m (full row rank).

8 Examples (explaining some of the examples in 6) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$ have rank 3,

but $\begin{bmatrix} 2 & 0 & 6 \\ 3 & 3 & -1 \\ 1 & 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 3 & 7 \end{bmatrix}$ have rank less than 3.

- 9 ****Theorem**** Fewer than m vectors cannot span \mathbf{R}^m .
- 10 ****Warning**** In \mathbf{R}^m , a set of m or more vectors does not necessarily span \mathbf{R}^m (but it might).
- 11 **Theorem** If $\mathbf{v}_1, \dots, \mathbf{v}_n$ are vectors in a vector space V , then the following conditions are equivalent:
- All $\mathbf{v}_i \neq \mathbf{0}$, and no \mathbf{v}_i is a linear combination of the others.
 - All $\mathbf{v}_i \neq \mathbf{0}$, and no \mathbf{v}_i is a linear combination of its “predecessors” $\mathbf{v}_1, \dots, \mathbf{v}_{i-1}$.
 - (For column vectors only) The matrix $[\mathbf{v}_1 \cdots \mathbf{v}_n]$ has rank n (i.e., full column rank, i.e., every column is a pivot column).
 - (For column vectors only) For the matrix $A = [\mathbf{v}_1 \cdots \mathbf{v}_n]$, $N(A)$ consists of just the zero vector $\mathbf{0}$. (That is, the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.)
 - The only scalars c_1, \dots, c_n which satisfy the equation $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ are the scalars $c_1 = \cdots = c_n = 0$.
- 12 **Definition** Vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ are **linearly independent** if and only if they satisfy one (hence all) of the conditions of the theorem just stated. (Correctly said: the **set** $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of vectors is linearly independent if and only if the conditions of the theorem are satisfied.) A set of vectors is **linearly dependent** if and only if it is not linearly independent.
- 13 **Examples**
- $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are linearly independent.
- $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(2, 3, 4)$ are not linearly independent.
- $(2, 3, 1)$, $(0, 3, 0)$ and $(6, -1, 3)$ are linearly dependent.
- $(1, 2, 3)$ and $(3, 5, 7)$ are linearly independent.
- The four matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ in $M_{2,2}$ are linearly independent.
- The three matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are linearly independent.
- For a given matrix A , the special solutions of $A\mathbf{x} = \mathbf{0}$ are linearly independent.
- For a given matrix A , the pivot columns of \mathbf{A} are linearly independent.
- For a given matrix A , the nonzero rows of its rref R are linearly independent.
- 14 ****Theorem**** In \mathbf{R}^m , a set consisting of n vectors must be linearly dependent if $n > m$.
- 15 ****Warning**** In \mathbf{R}^m , a set consisting of m or fewer vectors is not necessarily linearly independent, nor is it necessarily linearly dependent. Examples of both types occur.
- 16 ****Theorem**** For vectors $\mathbf{v}_1, \dots, \mathbf{v}_m$ in \mathbf{R}^m (NOTE the assumption of **just the “right number of vectors”!**), the following statements are equivalent:
- $\mathbf{v}_1, \dots, \mathbf{v}_m$ span \mathbf{R}^m .
 - $\mathbf{v}_1, \dots, \mathbf{v}_m$ are linearly independent.
- These conditions may be tested by seeing if the (square!) matrix $[\mathbf{v}_1, \dots, \mathbf{v}_m]$ is invertible.
- 17 **Definition** Vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ in a vector space V form a **basis of V** if and only if $\mathbf{v}_1, \dots, \mathbf{v}_n$ span V and $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly independent.

18 Examples

$(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ form a basis of \mathbf{R}^3 .

$(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(2, 3, 4)$ do not form a basis of \mathbf{R}^3 (or of any other vector space).

$(2, 3, 1)$, $(0, 3, 0)$ and $(6, -1, 3)$ do not form a basis of \mathbf{R}^3 (or of any other vector space).

$(1, 2, 3)$ and $(3, 5, 7)$ do not form a basis of \mathbf{R}^3 . However, if we let V be the vector space spanned by these two vectors (i.e. $V =$ the set of all linear combinations of these two vectors), then $(1, 2, 3)$ and $(3, 5, 7)$ do form a basis of V .

The four matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ form a basis of $M_{2,2}$.

The three matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ form a basis of $Sym_{2,2}$.

For a given matrix A , the special solutions of $A\mathbf{x} = 0$ form a basis of $N(A)$.

For a given matrix A , the pivot columns of \mathbf{A} form a basis of $C(A)$.

For a given matrix A , the nonzero rows of its rref R form a basis of $R(A)$.

19 Definition The dimension of a vector space V (notation: $\dim(V)$) is the number of vectors in a basis of V . The validity of this definition rests on two theorems:

20 Theorem Any vector space has a basis.

21 Theorem Any two bases of the same vector space have the same number of vectors.

22 Warning We have only defined “finite bases”. Some vector spaces have bases which consist of infinitely many vectors. We shall restrict attention to finite-dimensional vector spaces.

23 Theorem Any vector space of column vectors or of row vectors is finite-dimensional.

24 Examples

$\dim(\mathbf{R}^n) = n$. $\dim(M_{2,2}) = 4$. $\dim(M_{m,n}) = mn$. $\dim(Sym_{2,2}) = 3$.

$\dim(N(A)) =$ nullity of $A =$ number of free columns in $A = n - r$ (if A is $m \times n$ and has rank r).

$\dim(C(A)) = \dim(R(A)) =$ rank of $A =$ number of pivot columns in A .

25 Theorem If $\mathbf{v}_1, \dots, \mathbf{v}_n$ form a basis of the vector space V , then any vector \mathbf{v} in V has a unique expression $\mathbf{v} = c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n$ as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_n$.

26 Examples (See the bases mentioned in **18** above.)

A vector (a, b, c) in \mathbf{R}^3 has the unique expression

$$(a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1).$$

A matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in $M_{2,2}$ has the unique expression

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

A symmetric matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ in $Sym_{2,2}$ has the unique expression

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

27 Procedure If $\mathbf{v}_1, \dots, \mathbf{v}_n$ are vectors in a vector space V which span V , you can extract a basis of V from them as follows:

- a) If they are linearly independent, they are already a basis (STOP). If not, then one of them must be a linear combination of the others. Remove this **one**.
- b) The resulting set of $n - 1$ vectors still spans V . Repeat the procedure, starting with these $n - 1$ vectors. Continue repeating as necessary. (Do not remove more than one at a time! If you do, the resulting set might not span V .)

Alternate Procedure for column vectors: If $\mathbf{v}_1, \dots, \mathbf{v}_n$ are column vectors you can do it all at once by just selecting the pivot columns of the matrix $[\mathbf{v}_1, \dots, \mathbf{v}_n]$. The column space of this matrix is V , after all. Another alternate procedure is given below in **29**.

28 Procedure If $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly independent vectors in a (finite-dimensional) vector space V , you can supplement them with other vectors in V to form a basis as follows:

- a) If $\mathbf{v}_1, \dots, \mathbf{v}_n$ span V , they form a basis (STOP). If not, then some vector \mathbf{v} in V is not a linear combination of them. Find such a vector \mathbf{v} (sometimes easier said than done).
- b) The set $\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{v}$ is still linearly independent. Repeat the procedure, starting with these $n + 1$ vectors. Continue repeating as necessary. (Do not add more than one vector at a time! The resulting set might not be linearly independent.) The procedure produces ever-larger linearly independent sets, and must eventually stop before the number of vectors exceeds the dimension of V .

Alternate Procedure for column vectors: If $\mathbf{v}_1, \dots, \mathbf{v}_n$ are column vectors and you already know a basis (or even spanning set) $\mathbf{w}_1, \dots, \mathbf{w}_m$ of V you can do it all at once by using the pivot columns of the matrix $[\mathbf{v}_1 \cdots \mathbf{v}_n \mathbf{w}_1 \cdots \mathbf{w}_m]$.

29 Procedure If $\mathbf{v}_1, \dots, \mathbf{v}_n$ are column vectors which span V , and you are looking for a basis of V , you

can either use procedure **27** above, or you can form the matrix $A = \begin{bmatrix} \mathbf{v}_1^T \\ \cdots \\ \mathbf{v}_n^T \end{bmatrix}$ (i.e. convert $\mathbf{v}_1, \dots, \mathbf{v}_n$ to rows), reduce to rref R , and use the nonzero rows of R (converted back to columns).