Math 250-C1: Linear Algebra with MATLAB:Solutions Quiz 4, Oct. 1, 2003

(a) Determine whether the columns of the matrix A below are *linearly independent* or *linearly dependent*. Show work and give a reason for your answer.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

The column vectors are linearly independent if and only if the only scalars x_1, x_2, x_3 that satisfy

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 4 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

are $x_1 = x_2 = x_3 = 0$. (This is the same as saying that the only solution to $A\mathbf{x} = \mathbf{0}$ is the $\mathbf{x} = \mathbf{0}$.) If we write out this equation component by component, we get (writing them down in reverse order) $x_3 = 0$, $3x_3 = 0$, $2x_2 + 4x_3 = 0$ and $x_1 + 2x_2 + 3x_3 = 0$. The first two equations give $x_3 = 0$; since $x_3 = 0$, the second equation then implies that $x_2 = 0$, and since now x_2 and x_3 are both 0, the last equation implies $x_1 = 0$. Thus $x_1 = x_2 = x_3 = 0$ are the only scalars which make a linear combination of the columns equal to the zero vector, and hence the columns are linearly independent. (One can also solve this problem by showing that each column of the matrix is a pivot column, and this is how most of you did the problem. The solution given here just argues directly from the definition.)

(b) Find the parametric representation of the general solution to the system of equations $B\mathbf{x} = \mathbf{0}$, where

$$B = \begin{bmatrix} 1 & 2 & 5 & 8 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & 1 & 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

The solutions of $B\mathbf{x} = \mathbf{0}$ are the solutions of $R\mathbf{x} = \mathbf{0}$, where R is the RREF of B. Calculation shows that

$$R = \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

The free variables are x_3 and x_4 and the general solution is of the form (x_1, x_2, x_3, x_4) where $x_1 = -3x_3 - 4x_4$, and $x_2 = -x_3 - 2x_4$. The parametric representation of this general solution is

$$x_3 \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$