Math 250-C1 Quiz 11, Solutions, 2003

Let $A = \frac{1}{\sqrt{2}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Show that A is orthogonal and determine whether it is a rotation or reflection. If it is a rotation, determine the angle of rotation. If it is a reflection, either determine the angle the line of reflection makes with the x_1-axis, or give the equation for the line of reflection.

A calculation shows that $A^T A = I_2$, which shows that A is an orthogonal matrix. (Showing $|\det(A)| = 1$ does not show that A is orthogonal.)

A calculation shows that $\det(A) = -1$. Hence, A is a reflection; it has the form

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

for some θ. Inspection shows that $\theta = \pi/4$. This means that the line of reflection makes an angle of $\theta/2 = \pi/8$ with respect to the x_1-axis.

Alternatively, let the vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ point along the line of reflection. In this case we must have $A\mathbf{x} = \mathbf{x}$, that is, multiplication by A leaves $\mathbf{x}$ fixed (since it is in the line of reflection). The general solution of $A\mathbf{x} = \mathbf{x}$ is $x \begin{bmatrix} 1 \\ \sqrt{2}-1 \end{bmatrix}$, where x is a free variable. This means that the line of reflection is $y = (\sqrt{2}-1)x$. Note that $\sqrt{2}-1 = \tan(\pi/8)$.