Math 250-C1: Solutions to selected hand-in problems due Oct.6

Not all the assigned problems are solved here, but those that many students had trouble with are all included.

Problem 1. (Note that these solutions always give a concrete counterexample when the answer is false.)

section 1.2, 1b) False. The only linear combination of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ that equals $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ is

$$-\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Therefore one cannot choose positive scalars to get a linear combination of the two vectors with the same sum.

section 1.3, 1j) False. Here is an RREF of an augmented matrix with a zero row, but for which the corresponding system of linear equations is not consistent:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The second row corresponds to the equation $0x_1 + 0x_2 + 0x_3 = 1$ which has no solution.

section 1.4, 1b False. The matrix $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ can be reduced to the 2×2 identity matrix either by first dividing the first row by 2 and then interchanging rows or first interchanging rows and then dividing the second row by 2.

section 1.6, 1d False. I was perhaps overfussy when grading this. Most of you cited Theorem 1.5 which says that $\text{rank}(A) = m$ is the condition for consistency of $A\mathbf{x} = \mathbf{b}$ for each $\mathbf{b}$ in R^n. This is correct, but to be strictly logical, this does not exclude having $\text{rank}(A) = n$, so it is best to give an example where $\text{rank}(A) \neq n$. There are many possibilities; here is one: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

section 1.6, 1h False. The span of the single vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the same as the span of $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$, but the vectors are not the same.

section 1.7, 1k False. The zero vector in R^n is a single vector that forms a linearly dependent set.

Problem 3; section 1.3, 41 Let $[A\ \mathbf{b}]$ be the augmented matrix of a system of linear equations. Let $[R\ \mathbf{c}]$ be its RREF.

We first need to show that R is then the reduced row echelon form of A. This amounts to showing that if one removes the last column from an RREF matrix, what remains is an

RREF matrix. (It is this step that most of you forgot to include—it is pretty obvious from drawing a schematic RREF, but it needs to be shown to solve the problem.) Remember that an RREF is characterized by three conditions: first, every nonzero row lies above every zero row; second, the leading term in each row is 1 and lies to the right of the leading term in all previous rows; and third, in a column containing the leading term of a row, that leading term is the only non-zero entry in the column. Suppose now we remove the last column from R to get a new matrix R'. This will create a new zero row only if the row $[0\ 0\ \cdots\ 0\ 1]$ appears in R. If it does appear, this row will be the last non-zero row, since its leading term is in the last column. Thus, when the last column is removed, it will leave a new zero row following all non-zero rows and so no zero rows will be introcuced before non-zero rows. Thus the first property will still be true for R'. Since we remove only the last column, it will still be true that the leading term in each row will be 1. Finally, no new columns with leading terms are created in forming R' from R, so the columns of R' with leading terms still have the leading term as their only non-zero entry. Thus R' is again an RREF matrix.

Now since $[R\ \mathbf{c}]$ is obtained from $[A\ \mathbf{b}]$ by a sequence of elementary row operations, that same sequence of row operations will produce R from A. Since we have already shown that R is in RREF, it follows that R is the RREF of A.

Problem 5: section 1.4, 59 Let $\mathbf{u}$ and $\mathbf{v}$ be solutions to $A\mathbf{x} = \mathbf{b}$ and suppose $\mathbf{b} \neq \mathbf{0}$. Then

$$A(\mathbf{u} + \mathbf{v}) = \mathbf{Au} + \mathbf{Av} = \mathbf{b} + \mathbf{b} = \mathbf{2b} \neq \mathbf{b}.$$

Thus $\mathbf{u} + \mathbf{v}$ will **not** be a solution to $A\mathbf{x} = \mathbf{b}$. (It will however be a solution if $\mathbf{b} = \mathbf{0}$.)

Problem 6: section 1.6, 42. Prove that $\text{Span}(\{\mathbf{u}, \mathbf{v}\}) = \text{Span}(\{\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\})$.

Since $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are both in $\text{Span}(\{\mathbf{u}, \mathbf{v}\})$, Theorem 1.6 (b) implies that $\text{Span}(\{\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\})$ is contained in $\text{Span}(\{\mathbf{u}, \mathbf{v}\})$.

Now note that $\mathbf{u} = (\mathbf{1/2})(\mathbf{u} + \mathbf{v}) + (\mathbf{1/2})(\mathbf{u} - \mathbf{v})$. This means that $\mathbf{u}$ is in $\text{Span}(\{\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\})$. Likewise $\mathbf{v} = (\mathbf{1/2})(\mathbf{u} + \mathbf{v}) - (\mathbf{1/2})(\mathbf{u} - \mathbf{v})$, so $\mathbf{v}$ also is in $\text{Span}(\{\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\})$. Again from Theorem 1.6 (b), it follows that $\text{Span}(\{\mathbf{u}, \mathbf{v}\})$ is contained in $\text{Span}(\{\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\})$.

Combining the two inclusions proved in the preceding paragraphs implies $\text{Span}(\{\mathbf{u}, \mathbf{v}\}) = \text{Span}(\{\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\})$.

Problem 7: section 1.7, 38. Let $\mathcal{S} = \{\mathbf{u_1}, \ldots, \mathbf{u_k}\}$ be a linearly independent set of vectors. Suppose that $\{\mathbf{v_1}, \ldots, \mathbf{v}_\ell\}$ is another set of vectors that spans $\text{Span}(\mathcal{S})$. We want to show $\ell \geq k$.

If $\{\mathbf{v_1}, \ldots, \mathbf{v}_\ell\}$ is linearly dependent, we can remove one of the vectors which is a linear combination of the others without changing the span. Keep removing vectors in this way until only a linearly independent subset $\{\mathbf{w_1}, \ldots, \mathbf{w_m}\}$ is left with $\text{Span}(\{\mathbf{w_1}, \ldots, \mathbf{w_m}\}) = \text{Span}(\mathcal{S})$. Because $\{\mathbf{w_1}, \ldots, \mathbf{w_m}\}$ was created by removing vectors from $\{\mathbf{v_1}, \ldots, \mathbf{v}_\ell\}$, m, the number of vectors in $\{\mathbf{w_1}, \ldots, \mathbf{w_m}\}$, is less than or equal to ℓ.

By Theorem 1.9, every subset of $\text{Span}(\{\mathbf{w_1}, \ldots, \mathbf{w_m}\})$ that contains more than m vectors is linearly dependent. Since $\mathcal{S}$ is a subset of $\text{Span}(\mathcal{S}) = \text{Span}(\{\mathbf{w_1}, \ldots, \mathbf{w_m}\})$ and $\mathcal{S}$ is linearly independent and has k vectors, it follows that $k \leq m$. Since we showed above that $m \leq \ell$, we derive that $k \leq \ell$, as well.