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1. Compute the determinant of the following matrix **using cofactor expansions.**

$$\begin{bmatrix} 0 & 5 & 3 & 1 \\ 1 & -2 & -1 & 1 \\ -1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

Expand along the last row: $\det(A) = (-1)^{(4+2)} 2 \det \begin{bmatrix} 0 & 3 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$.

Now expand along the first row of the 3×3 matrix:

$$\det(A) = 2\left((-1)^3 3 \det \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} + \det \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right) = 2(-3+0) = -6.$$

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2. In each case determine whether the given set of vectors is a subspace. Supply a reason for your answer.

a) $V = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathcal{R}^3 \,;\, v_2 + 5v_3 = 0 \right\}$ is a subspace, because it is the null space of the matrix $A = \begin{bmatrix} 0 & 1 & 5 \end{bmatrix}$. For an alternate and more complete explanation, check that V is closed under the operation of taking linear combinations. Suppose that $\mathbf{v}$ and $\mathbf{u}$ are both in V and let $\mathbf{w} = a\mathbf{v} + b\mathbf{u}$. To check that $\mathbf{w}$ is in V, we must check $w_2 + 5w_3 = 0$. But $w_2 + 5w_3 = (av_2 + bu_2) + 5(av_3 + bu_3) = a(v_2 + 5v_3) + b(u_2 + 5u_3)$. Now $v_2 + 5v_3 = 0$ and $u_2 + 5u_3 = 0$ since $\mathbf{v}$ and $\mathbf{u}$ are in V. Hence $w_2 + 5w_3 = 0$.

b) $V = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathcal{R}^3 \,;\, v_1 - v_2^2 = 0 \right\}$ is not a vector space. Consider the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Check that $\mathbf{v}$ is in V but $2\mathbf{v}$ is not, so V is not closed under the operation of taking linear combinations.

3. Let A be the matrix $\begin{bmatrix} 1 & -3 & 1 & 1 \\ -3 & 9 & -1 & -7 \\ 2 & -6 & -1 & 8 \\ -1 & 3 & -5 & 7 \end{bmatrix}$.

The reduced row echelon form of A is $\begin{bmatrix} 1 & -3 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

a) What is the dimension of Col(A)? $\dim \mathrm{Col}(A) = 2$.
The dimension of Row(A)? $\dim \mathrm{Row}(A) = 2$ Of Null(A)? $\dim \mathrm{Null}(A) = 2$.

b) Find a basis of Row(A). A basis of Row(A) is given by the non-zero rows of the RREF of A. Thus $\{\begin{bmatrix} 1 & -3 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & -2 \end{bmatrix}\}$. The first two rows of A are also a basis of the row space, and many students gave that as an answer. Can you say why it is a basis?

c) Find a basis of Col(A). The pivot columns of A form a basis of Col(A). Thus a basis is:

$$\left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ -5 \end{bmatrix} \right\}$$

d) Find a basis of Null(A). By solving $R\mathbf{x} = \mathbf{0}$, where R is the RREF of A, the general solution of $A\mathbf{x} = \mathbf{0}$ is the span of $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$. The vectors in this set are linearly independent and so form a basis.

e) Is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 3 \\ 10 \\ 5 \end{bmatrix} \right\}$ a basis of Null(A)? Give an argument for your answer. Yes, it is a basis. Call the two vectors $\mathbf{v}$ an $\mathbf{u}$. We need to check: (i) $\mathbf{v}$ and $\mathbf{u}$ are linearly indepedent; (ii) $A\mathbf{v} = \mathbf{0}$ and $A\mathbf{u} = \mathbf{0}$ so that we know $\mathbf{v}$ and $\mathbf{u}$ are in Null(A); (iii) $2 = \dim \mathrm{Null}(A)$. (i) is easy because the first entry of $\mathbf{v}$ is 0, while the first entry of $\mathbf{u}$ is not. (ii) we can check by direct calculation. (iii) is a consequence of part (a).

f) Is $\left\{ \begin{bmatrix} -3 \\ 9 \\ -6 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 6 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 5 \end{bmatrix} \right\}$ a basis of Col(A)? Give a reason for your answer.

It cannot be a basis. The dimension of Col(A) is 2 so any basis must have 2 vectors.

4 Find the LU decomposition of $A = \begin{bmatrix} -2 & 0 & 5 \\ 2 & 2 & -2 \\ -4 & 6 & 20 \end{bmatrix}$.

Answer: $L = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$ $\quad U = \begin{bmatrix} -2 & 0 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

5. Find the inverse of $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ -3 & 4 & 2 \end{bmatrix}$.

The inverse of $A^{-1} = A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1/2 & 0 \\ 7/2 & -1 & 1/2 \end{bmatrix}$.

6. Let A be expressed as the product of elementary matrices,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

a) Calculate $\det(A)$. Express the product as $A = E_1E_2E_3$. Then $\det(A) = \det(E_1)\det(E_2)\det(E_3) = 1(-2)(-1) = 2$.
b) Express A^{-1} as a product of elementary matrices. Again, expressing A as $A = E_1E_2E_3$,

$$A^{-1} = E_3^{-1}E_2^{-1}E_1^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

7. Suppose that $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} = 5.$

Find the determinants of the following matrices. Indicate briefly your reasoning.

a) $\det \begin{bmatrix} a+g & b+h & c+j \\ d & e & f \\ g & h & j \end{bmatrix} = 5$, because replacing a row by that row plus a multiple of another does not change the determinant.

b) $\det \begin{bmatrix} 2a & 2b & 2c \\ 2d & 2e & 2f \\ g & h & j \end{bmatrix} = 2 \cdot 2 \cdot 5 = 20$, because multiplying a row by a scalar c changes the determinant by the same multiplicative factor c.

c) $\det \begin{bmatrix} d & a & g \\ e & b & h \\ f & c & j \end{bmatrix} = -5$, because the matrix here is obtained by first taking the transpose, which does not change the determinant, and then interchanging columns, which changes the determinant by a factor of -1.

8. In each case, determine whether the statement is true or false. Explain each answer, by using an example, or citing a theorem or fact, or giving an argument.

a) Every matrix A has a decomposition of the form $A = LU$ where L is a square, unit lower triangular matrix and U is upper triangular.

False. If a row interchange is required in finding a row echelon form of A, then A does not have an LU-decomposition.

b) If V is a subspace of $\mathcal{R}^n$, every basis of V has the same number of vectors.

True. It is shown in the book that every basis of a subspace has the same number of elements and this fact allows one to define the dimension of a subspace.

c) If A is a 3×3 matrix and $A^T = -A$, then A has no inverse.

True.
If A is 3×3, $\det(-A) = -\det(A)$. Thus $-\det(A) = \det(-A) = \det(A^T) = \det(A)$, which implies $\det(A) = 0$. A matrix with zero determinant is not invertible.