

1. Write the parametric representation of the general solution to the system of linear equations

$$\begin{aligned}x_1 - x_2 - 2x_3 - 4x_4 &= 0 \\x_1 - x_2 - x_3 - x_4 &= 0 \\-2x_1 + 2x_2 + 6x_3 + 14x_4 &= 0\end{aligned}$$

Row reduce the coefficient matrix  $\begin{bmatrix} 1 & -1 & -2 & -4 \\ 1 & -1 & -1 & -1 \\ -2 & 2 & 6 & 14 \end{bmatrix}$  to RREF form. The result is

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solve  $R\mathbf{x} = \mathbf{0}$ . The solution is  $x_1 = x_2 - 2x_4$ ,  $x_2$  is free,  $x_3 = -3x_4$ ,  $x_4$  is free. The parametric expression of the general solution is

$$x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -3 \\ 0 \end{bmatrix}$$

2. For which value(s) of  $r$  is the set of vectors  $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ r \end{bmatrix} \right\}$  linearly dependent? Linearly independent?

The set is linearly independent if and only if the rank of the matrix  $A$  is 3, where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 4 \\ 3 & 1 & r \end{bmatrix}$$

$A$  can be reduced to the row echelon form

$$U = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & r - 7 \end{bmatrix}$$

(We could reduce to RREF, but we don't need to go all the way to the RREF to calculate the rank.) The rank is 3 if and only if  $r \neq 7$ . Thus the set of vectors is linearly dependent if  $r = 7$  and linearly independent if  $r \neq 7$ .

3. In each case, determine if the statement is true or false. Explain your answer, for instance, with a brief argument or an example.

(a) There exists a  $5 \times 8$  matrix with rank 4 and nullity 2.

False. The nullity of a  $5 \times 8$  matrix with rank 4 is  $8 - 4 = 4$ .

(b) A system of linear equations with fewer equations than unknowns may not have a solution.

True. The system of equations  $x_1 + x_2 + x_3 = 0$ ,  $2x_1 + 2x_2 + 2x_3 = 1$  has no solution.

(c) If the first column of a matrix in reduced row echelon form is removed, the matrix that remains is in reduced row echelon form.

False. The matrix  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  is in RREF, but the matrix obtained by removing the first column is  $\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ , which is not in RREF

(d) There is no choice of constants  $a$ ,  $b$ ,  $c$ , and  $d$ , such that the set  $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right\}$  is linearly independent.

True.  $\mathbb{R}^4$  is spanned by four standard vectors. According to Theorem 1.9, any set of vectors in  $\mathbb{R}^4$  which has more than 4 vectors must be linearly dependent. (Other, more direct reasoning was also perfectly acceptable as an answer.)

4. Let  $A$  be the matrix  $\begin{bmatrix} 1 & -3 & 1 & 1 & 2 \\ -3 & 9 & -1 & -7 & 0 \\ 2 & -6 & -1 & 8 & 2 \\ -1 & 3 & -5 & 7 & -12 \end{bmatrix}$ .

The reduced row echelon form of  $A$  is  $\begin{bmatrix} 1 & -3 & 0 & 3 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ .

Answer the following questions about the given matrix  $A$ . Explain briefly how you got your answer.

a) Is the system of linear equations  $A\mathbf{x} = \mathbf{b}$  consistent for every  $\mathbf{b}$  in  $\mathbb{R}^4$ ?

No. The rank of  $A$  is 3 and  $A\mathbf{x} = \mathbf{b}$  is consistent for every  $\mathbf{b}$  in  $\mathbb{R}^4$  if and only if the rank of  $A$  is 4.

b) What are the pivot columns of  $A$  ?

The pivot columns of  $A$  are the columns of  $A$  which correspond to the columns of the RREF of  $A$  containing leading terms of rows. These are columns 1, 3, and 5.

c) If  $A\mathbf{x} = \mathbf{b}$  has a solution, is that solution unique?

No. Since the nullity of  $A$  is 1, which is greater than 0, if  $A\mathbf{x} = \mathbf{b}$  has a solution, there will be an infinite number of solutions.

5. Calculate  $A\mathbf{c}$  where  $A = \begin{bmatrix} 1 & -3 & 4 \\ 3 & -7 & 2 \\ -1 & 4 & -9 \end{bmatrix}$  and  $\mathbf{c} = \begin{bmatrix} 11 \\ 5 \\ 1 \end{bmatrix}$ .

From your answer, can you conclude anything about the columns of  $A$  ?

A calculation shows that  $A\mathbf{c} = \mathbf{0}$  and this implies that the columns of  $A$  form a linearly independent set of vectors.

6. Assume that  $\mathbf{u}$  and  $\mathbf{v}$  are linearly independent vectors. Show that  $\mathbf{u} + \mathbf{v}$  and  $\mathbf{u} - 2\mathbf{v}$  are also linearly independent.

There was some confusion of interpretation of this problem. Saying that  $\mathbf{u}$  and  $\mathbf{v}$  are linearly independent, is the same as saying that the set  $\{\mathbf{u}, \mathbf{v}\}$  is linearly independent.

To show  $\mathbf{u} + \mathbf{v}$  and  $\mathbf{u} - 2\mathbf{v}$  are linearly independent (that is, the set  $\{\mathbf{u} + \mathbf{v}, \mathbf{u} - 2\mathbf{v}\}$  is linearly independent), one must show that the only solution to  $c_1(\mathbf{u} + \mathbf{v}) + c_2(\mathbf{u} - 2\mathbf{v}) = \mathbf{0}$  is  $c_1 = 0, c_2 = 0$ . However, if  $\mathbf{0} = c_1(\mathbf{u} + \mathbf{v}) + c_2(\mathbf{u} - 2\mathbf{v})$ , then  $\mathbf{0} = (c_1 + c_2)\mathbf{u} + (c_1 - 2c_2)\mathbf{v}$ . But since  $\mathbf{u}$  and  $\mathbf{v}$  are linearly independent, this can happen only if  $c_1 + c_2 = 0$  and  $c_1 - 2c_2 = 0$ . One can check that the only solution to this set of equations is  $c_1 = 0$  and  $c_2 = 0$ .

7. Find the full  $4 \times 4$  matrix  $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \mathbf{a}_4]$  (where  $\mathbf{a}_i$  is column  $i$  of  $A$ ), given

that  $\mathbf{a}_2 = \begin{bmatrix} 3 \\ 2 \\ -1 \\ 1 \end{bmatrix}$ ,  $\mathbf{a}_3 = \begin{bmatrix} 2 \\ 1 \\ -4 \\ -2 \end{bmatrix}$ , and that the reduced row echelon form of  $A$  is

$$R = \begin{bmatrix} 1 & 3 & 0 & -2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let  $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4$  be the columns of  $R$ . By the linear correspondence property, the linear relations among the columns of  $R$  are the same as those among the columns of  $A$ . Since  $\mathbf{r}_2 = 3\mathbf{r}_1$ ,  $\mathbf{a}_2 = 3\mathbf{a}_1$  and hence

$$b\mathbf{a}_1 = \frac{1}{3}\mathbf{a}_2 = \frac{1}{3} \begin{bmatrix} 3 \\ 2 \\ -1 \\ 1 \end{bmatrix}.$$

Since  $\mathbf{r}_4 = -2\mathbf{r}_1 + 5\mathbf{r}_3$ ,

$$\mathbf{a}_4 = -2\mathbf{a}_1 + 5\mathbf{a}_3 = -2 \begin{bmatrix} 1 \\ 2/3 \\ -1/3 \\ 1/3 \end{bmatrix} + 5 \begin{bmatrix} 2 \\ 1 \\ -4 \\ -2 \end{bmatrix} = \begin{bmatrix} 8 \\ 11/3 \\ -58/3 \\ -32/3 \end{bmatrix}$$

Therefore

$$A = \begin{bmatrix} 1 & 3 & 2 & 8 \\ 2/3 & 2 & 1 & 11/3 \\ -1/3 & -1 & -4 & -58/3 \\ 1/3 & 1 & -2 & -32/3 \end{bmatrix}.$$

8. Find a set of vectors that is as small as possible and has the same span as

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Show your work and your method.

Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 1 & 0 & -1 & 3 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

The pivot columns of  $A$  will form a smallest spanning set for the span of all the columns of  $A$ . Row reduction transforms  $A$  into its RREF,

$$R = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Columns 1, 2, and 3 are the pivot columns. Thus the smallest spanning set is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\}.$$