

LAB 6: Eigenvalues and Eigenvectors

In this lab you will use MATLAB to study three topics:

- The geometric meaning of eigenvalues and eigenvectors of a matrix
- Determination of eigenvalues and eigenvectors using the characteristic polynomial of a matrix
- The special properties of eigenvalues and eigenvectors for a symmetric matrix

Reading from Textbook: In connection with this Lab, read through Sections 8.1, 8.2, and 8.3 of the text and work the suggested problems for each section.

Script Files: For this lab you will need the the m-file `rmat.m` from Lab 3.

Lab Write-up: You should open a diary file at the beginning of each MATLAB session (see Lab 1 for details). Be sure to answer all the questions in the lab assignment.

Random Seed: When you start your MATLAB session, initialize the random number generator by typing

```
rand('seed', abcd)
```

where $abcd$ are the last four digits of your Student ID number. This will ensure that you generate your own particular random vectors and matrices.

BE SURE TO INCLUDE THIS LINE IN YOUR LAB WRITE-UP

The lab report that you hand in must be your own work. The following problems all use randomly generated matrices and vectors, so the matrices and vectors in your lab report will not be the same as those of other students doing the lab. Sharing of lab report files is not allowed in this course.

Question 1. Graphic Demonstration of Eigenvectors and Eigenvalues

(a) Type `eigshow` at the MATLAB prompt. A graphics window should open. Underneath the graph the statement

```
Make A*x parallel to x
```

should appear (if it does not, then click on the `eig` button to get this statement).

Click on the pull-down bar above the graph and select the matrix $[1 \ 3; 4 \ 2]/4$. Move the cursor onto the vector \mathbf{x} , and make \mathbf{x} go around a full circle. The transformed vector $A\mathbf{x}$ then moves around an ellipse. Search for the *special directions* where $A\mathbf{x}$ and \mathbf{x} lie on a straight line. When \mathbf{x} points in one of these directions, it is an *eigenvector* of the matrix A (the word *eigen* means *special* in German). For \mathbf{x} pointing in these *special directions*, $A\mathbf{x} = \lambda\mathbf{x}$, where λ is an *eigenvalue* of A . Since \mathbf{x} is a unit vector, the length of $A\mathbf{x}$ is $|\lambda|$. If $A\mathbf{x}$ points in the same direction as \mathbf{x} , then $\lambda > 0$. If $A\mathbf{x}$ points in the opposite direction to \mathbf{x} , then $\lambda < 0$.

From your graphical experimentation answer the following questions (no algebraic calculations needed):

1. How many positive eigenvalues does A have?
2. How many negative eigenvalues does A have?
3. What are the (approximate) numerical values of the eigenvalues?
4. Are the eigenvectors on the axes of the ellipse traced out by $A\mathbf{x}$?

(Don't try to print the eigshow window.)

(b) At the MATLAB prompt type

```
A = [1, 3; 3, 2]/4 ; eigshow(A)
```

Now the new matrix A should appear in the eigshow window above the graph. Move \mathbf{x} around the circle with the cursor, as in part (a), and observe the ellipse traced out by $A\mathbf{x}$. From your graphical experimentation answer the following questions (no algebraic calculations needed):

1. How many positive eigenvalues does A have?
2. How many negative eigenvalues does A have?
3. What are the (approximate) numerical values of the eigenvalues?
4. Are the eigenvectors on the axes of the ellipse traced out by $A\mathbf{x}$?

(c) Finally click on pull-down matrix selection bar and select $[3 \ 1; -2 \ 4]/4$. Move \mathbf{x} around the circle with the cursor. Are there any directions where $A\mathbf{x}$ is parallel to \mathbf{x} ? Does A have any *real* eigenvalues?

Question 2. Calculation of Eigenvalues and Eigenvectors

(a) **Characteristic Polynomial** At the MATLAB prompt type $A = [1 \ 3; 4 \ 2]/4$ (this is the matrix in part (a) of Question #1). The eigenvalues of A are the roots of the *characteristic polynomial* of A . Use the MATLAB to calculate its characteristic polynomial $p(s)$ by

```
syms s; I = eye(2); p = det(A - s*I)
```

(you must be running MATLAB in an environment where the symbolic toolbox is installed to do this; otherwise calculate $p(s)$ by hand). Verify that the constant term in the polynomial $p(s)$ is $\det(A)$. Verify that the coefficient of s is $-\text{trace}(A)$ (the *trace* of a matrix is the sum of the diagonal entries). Use the MATLAB command `solve(p)` to get the roots of $p(s)$ (the eigenvalues of A). Compare these values with your graphical estimates for the eigenvalues from Question #1(a).

Now at the MATLAB prompt type $A = [3 \ 1; -2 \ 4]/4$ (this is the matrix in part (c) of Question #1). Calculate the characteristic polynomial $p(s)$ of A as before (use the \uparrow key) and find its roots. Are the roots real? How does this explain what you observed Question #1(c)?

(b) **Eigenvectors and Eigenvalues:** Generate a random 3×3 integer matrix $A = \text{rmat}(3,3)$. Use the MATLAB command

```
[P D] = eig(A)
```

to generate a matrix P and a diagonal matrix D . Use MATLAB to define

```
x1 = P(:,1), x2 = P(:,2), x3 = P(:,3)
```

(the columns of P) Calculate $A*x_1$, $A*x_2$, $A*x_3$. Then calculate $D(1,1)*x_1$, $D(2,2)*x_2$, $D(3,3)*x_3$. What does this tell you about the eigenvalues and eigenvectors of A ? (See Theorem 8.4, page 357.)

(c) Diagonalization: Let A, P, D be as in part (b). Verify by MATLAB that $A = P*D*inv(P)$. How can you use this to express A^5 and A^{10} in terms of D and P ? Explain and verify your answers using MATLAB.

Question 3. Eigenvalues and Eigenvectors for Symmetric Matrices

(a) Orthogonality of Eigenvectors: Define the matrix

$$A = [1, 3; 3, 2]/4$$

(as in Question #1(b)). Use MATLAB to calculate eigenvalues and eigenvectors of A by

$$[Q, E] = \text{eig}(A)$$

(the columns of Q are the normalized eigenvectors and the diagonal entries of E are the eigenvalues). Compare the calculated eigenvalues with your graphical estimates for the eigenvalues from Question #1(b).

Use MATLAB to verify that Q is an orthogonal matrix. What does this tell you about the angle between the two eigenvectors? How does this relate to your graphical experimentation in Question #1(b)? Verify that $A = Q * E * Q^T$.

(b) 3×3 Symmetric Matrix: Generate a random 3×3 symmetric matrix A by

$$B = \text{fix}(5*\text{rand}(3)); A = B + B'$$

Since A is a symmetric matrix, all its eigenvalues will be real. Have MATLAB find the eigenvalues and eigenvectors of A by

$$[Q, E] = \text{eig}(A)$$

(the columns of Q are the normalized eigenvectors and the diagonal entries of E are the eigenvalues). Use MATLAB to verify that Q is an orthogonal matrix. What does this tell you about the dot products between the eigenvectors? Verify that $A = Q * E * Q^T$ (see Theorem 8.9 on page 369).

Optional Extra-Credit Question: Iterative Method for Eigenvectors

The numerical computation of eigenvalues and eigenvectors by MATLAB does not use the methods in the text (calculate the characteristic polynomial, find its roots, and then solve homogeneous systems of equations to get the eigenvectors). Instead, the eigenvectors are found first by iterative methods, and the eigenvalues are then determined from the eigenvectors. In this question you will see the simplest example of this procedure, the *power method*.

(a) Dominant Eigenvector: Generate a random 2×2 real symmetric matrix A and a random initial unit vector u by

$$B = \text{rmat}(2,2), \quad A = B + B', \quad v = \text{rvect}(2), \quad u = v/\text{norm}(v)$$

Plot the vector u by the command

$$\text{plot}([0,u(1)], [0, u(2)]), \quad \text{hold on}$$

Since A is symmetric, it has two real eigenvalues λ_1, λ_2 , listed so that $|\lambda_1| \geq |\lambda_2|$. The *power method* uses \mathbf{u} as the initial guess for the eigenvector of A corresponding to λ_1 . Generate successive approximations to the eigenvector and plot them by the commands

```
v = A*u,    u = v/norm(v),    plot([0, u(1)], [0, u(2)])
```

Use the \uparrow key to repeat these commands until the first three decimal places of the entries in \mathbf{u} don't change under iteration. (Be sure that you are using `format short` to display the results.) How many iterations were needed to achieve this numerical convergence? What happens to the directions of the unit vectors plotted in the graphics windows during the iterations? Print the graphics window at the end of the iteration, and include the graph in your lab write-up.

(b) Dominant Eigenvalue: Let \mathbf{u} and \mathbf{v} be the vectors you obtained in part (a) at the end of the iterations. Calculate

```
r = norm(v), r*u, A*u
```

Explain why this shows that \mathbf{u} is an approximate eigenvector for A . What is the corresponding eigenvalue?

(c) Complete Set of Eigenvectors: Now use MATLAB to calculate both eigenvectors and eigenvalues of A by

```
[Q, E] = eig(A)
```

Which eigenvalue of A have you obtained in (b), the larger or the smaller? Does your vector \mathbf{u} agree with the MATLAB calculation of this eigenvector for this eigenvalue (to three decimal places)?

(d) Algebraic Description of Algorithm: To show why the algorithm in part (a) is called the *power method*, carry out the following hand calculation: Write \mathbf{v}_0 and \mathbf{u}_0 for the initial random vectors. Let \mathbf{v}_k and \mathbf{u}_k be the vectors for the k^{th} iteration. For example,

$$\mathbf{v}_1 = A\mathbf{v}_0/\|\mathbf{v}_0\|, \quad \mathbf{u}_1 = A\mathbf{v}_0/\|A\mathbf{v}_0\|$$

Find similar formulas for \mathbf{v}_k and \mathbf{u}_k for $k = 2, 3$. Write everything in terms of \mathbf{v}_0 and powers of A . (Don't do any numerical calculation; give the formulas algebraically.)

Note: MATLAB uses a faster and more accurate iterative method called the *QR Algorithm* to find eigenvectors and eigenvalues. It is based on the *QR* factorization of the matrix A .

Final Editing of Lab Write-up: After you have worked through all the parts of the lab assignment, you will need to edit your diary file. Remove all errors and other material that is not directly related to the questions. Your write-up should only contain the required MATLAB calculations and the answers to the questions. Preview the document before printing and remove unnecessary page breaks and blank space.