To review for the final exam, use your homework assignments, your midterms, and the two previous review sheets supplied by the department, which cover the material through §10.4.

The final examination is a common examination, set by the department, and graded by the department. It covers the work of the entire term. Calculators and notes are not permitted. A formula sheet will be supplied.

The final examination for Math 152 will be given on Thursday, December 16th, 2010 from 4:00 pm to 7:00 pm. Your final exam schedule can be found at http://finalexams.rutgers.edu/.

Additional Review Problems on Sequences and Series

1. Interval of Convergence
Find the precise interval of convergence for each of the series

(a) \( \sum_{n=1}^{\infty} \frac{(x-3)^n}{(n+1)2^n} \)  
(b) \( \sum_{n=1}^{\infty} \frac{(-1)^n(x+3)^n}{\sqrt{n+1}} \)

Check the endpoints, and specify whether the convergence is conditional or absolute.

2. Interval of Convergence
If there is a power series with the specified interval of convergence, give an example. If not, explain why not.

(a) \((-2, 1)\)  
(b) \([0, \infty)\)  
(c) \((-1, 1]\)

3. Tests for Convergence and Divergence
State which tests are used in each case to determine whether the series converges, and what the conclusion is.

(a) \( \sum_{n=7}^{\infty} \frac{(-1)^n(n+3)}{n+6} \)  
(b) \( \sum_{n=2}^{\infty} \sqrt{\frac{n^2 + 1}{n^5 - n^4 - 2}} \)  
(c) \( \sum_{n=1}^{\infty} \frac{\sin(1/n)}{n} \)

(d) \( \sum_{n=3}^{\infty} \frac{\sin^2(e^n)}{n^{4/3}} \)  
(e) \( \sum_{n=1}^{\infty} \frac{n^n}{3^n n!} \)  
(f) \( \sum_{n=1}^{\infty} \frac{3n^2 - n - 1}{5n^3 + n^2 + 5} \)

(g) \( \sum_{n=1}^{\infty} \frac{\sin(1/n^2)}{n} \)  
(h) \( \sum_{n=1}^{\infty} \frac{3^n + n}{4^n - 3^n} \)  
(i) \( \sum_{n=1}^{\infty} \frac{1}{n(\ln n)^4} \)

(j) \( \sum_{n=1}^{\infty} \frac{(2n)!/n!}{(3n)!} \)  
(k) \( \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} \)  
(l) \( \sum_{n=3}^{\infty} \frac{\cos(n^3)}{n(n\ln n)^2} \)

(m) \( \sum_{n=2}^{\infty} \frac{7^n n^7}{2^{2n-3}} \)  
(n) \( \sum_{n=1}^{\infty} \frac{(n!)^2(n)^n}{(2n)!} \)  
(o) \( \sum_{n=1}^{\infty} \frac{(-1)^{n+3}}{\sqrt{n + 3}} \)
   (a) The “hyperbolic cosine” (cosh(x)) is defined by cosh(x) = \( \frac{e^x + e^{-x}}{2} \).
   Find its Maclaurin expansion. How is this series related to the Maclaurin expansion of cos(x)?
   (b) Calculate the first 3 nonzero terms of the Maclaurin series for \( \tan x \).
   (c) Calculate the first 5 nonzero terms of the Maclaurin series for \( x^3 e^{-x^2} \).

5. Evaluation of power series
   Let \( f(x) \) be defined by \( f(x) = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!} \).
   (a) Show that \( f'(x) = 2f(x) \)
   (b) Find an explicit formula for \( f(x) \).

6. The Binomial Series
   Find the first 4 nonzero terms of the Maclaurin expansion of \( f(x) = \sqrt{4 + x^2} \).

7. Convergence of Power Series
   Use the Maclaurin expansion for \( e^x \) to show that
   \[ e = \sum_{n=0}^{\infty} \frac{1}{n!} \]
   What theorems about series are involved in justifying this claim?

8. Evaluating Power Series
   Find the Taylor expansion of \( \frac{1}{(1 - x)^2} \) by the following three methods.
   (a) Apply the Binomial Theorem.
   (b) Square the series for \( \frac{1}{1-x} \).
   (c) Differentiate the series for \( \frac{1}{1-x} \).
   Rank the methods according to your preference (for this particular case).

9. Summation Formulas
   Calculate \( \sum_{n=1}^{\infty} \frac{n}{2^n} \) exactly:
   (a) Obtain a simple formula for the sum \( \sum_{n=1}^{\infty} nx^n \).
   (b) Check that \( x = 1/2 \) is in the interval of convergence, and then take \( x = 1/2 \).

(a) Use the Maclaurin expansion of degree 3 for sin \( x \) to estimate \( \int_0^{1/2} \frac{\sin(\sqrt{x})}{\sqrt{x}} \, dx \).

(b) Give an estimate for the resulting error.

(c) How many (non-zero) terms of the expansion would be needed to meet an error tolerance of \( 10^{-6} \)?

11. Limits

Find \( \lim_{n \to \infty} \left[ \frac{\sin \left( \frac{1}{n} \right) - \left( \frac{1}{n} \right)}{\left( e^{\frac{1}{n}} - 1 \right) \left( 1 - \cos \left( \frac{1}{n} \right) \right)} \right] \) using the Maclaurin expansions for \( \sin, \cos, \) and \( e^x \).

12. \( p \)-series with \( p = 2 \)

The mathematician Leonhard Euler showed in 1734 that the sum \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) is equal to \( \frac{\pi^2}{6} \) (this took him 4 years). Using this fact, calculate the following related sums:

(a) \( A = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \)  
(b) \( B = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \)  
(c) \( C = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \)

Hint: The easiest is (b).

How could each of these series be used to calculate \( \pi \)?