Notes on the Review Problems for Midterm 2

(1)(a) The partial fractions expansion \( \frac{1}{(x - 3)(x - 4)} = \frac{1}{x - 4} - \frac{1}{x - 3} \) gives

\[
\int_0^\infty \frac{dx}{(x - 3)(x - 4)} = \lim_{b \to \infty} \left( \ln |x - 4| - \ln |x - 3| \right) \bigg|_0^b = \lim_{b \to \infty} \left( \ln \left( \frac{b - 4}{b - 3} \right) + \ln 2 \right) = \ln 2.
\]

(1)(b) L'Hôpital's Rule lets us write

\[
\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} -x = 0.
\]

This can be rewritten \( \lim_{a \to 0^+} a \ln a = 0 \). Now we get

\[
\int_0^1 \ln x \, dx = \lim_{a \to 0^+} \left( \int_a^1 \ln x \, dx \right) = \lim_{a \to 0^+} (x \ln x - x) \bigg|_a^1 = -1.
\]

(1)(c) Integration by parts gives \( \int x^n e^{-x} \, dx = -x^n e^{-x} + n \int x^{n-1} e^{-x} \, dx \). Now the fact \( \lim_{b \to \infty} b^n e^{-b} = 0 \) (which we get from l'Hôpital's Rule) lets us write

\[
\int_0^{\infty} x^n e^{-x} \, dx = n \int_0^{\infty} x^{n-1} e^{-x} \, dx.
\]

Therefore, \( \int_0^{\infty} x^3 e^{-x} \, dx = 3 \int_0^{\infty} x^2 e^{-x} \, dx = 6 \int_0^{\infty} x e^{-x} \, dx = 6 \int_0^{\infty} e^{-x} \, dx = 6. \)

(1)(d) The substitution \( x = 3 \tan \theta \) gives

\[
\int \frac{dx}{9 + x^2} = \int \frac{3 \sec^2 \theta \, d\theta}{9(1 + \tan^2 \theta)} = \int \frac{d\theta}{3} = \frac{\theta}{3} + C = \frac{1}{3} \tan^{-1} \left( \frac{x}{3} \right) + C.
\]

Therefore, \( \int_{-\infty}^{\infty} \frac{dx}{9 + x^2} = \lim_{b \to \infty} \left( \frac{1}{3} \tan^{-1} \left( \frac{b}{3} \right) - \frac{1}{3} \tan^{-1} \left( \frac{-b}{3} \right) \right) = \frac{1}{3} \left( \frac{\pi}{2} - \left( -\frac{\pi}{2} \right) \right) = \frac{\pi}{3}. \)

(2)(a) For \( x \geq 7 \) we know \( 0 < x - |\cos x| \leq x \), hence \( \frac{1}{x - |\cos x|} \geq \frac{1}{x} > 0 \). The divergence of \( \int_7^{\infty} \frac{dx}{x} \) implies the divergence of \( \int_7^{\infty} \frac{dx}{x - |\cos x|} \).

(2)(b) For \( x \geq 5 \) we know \( 0 < \frac{1}{e^{x^2}} < \frac{1}{e^x} \). The convergence of \( \int_5^{\infty} \frac{dx}{e^x} \) (which is just \( \int_5^{\infty} e^{-x} \, dx \)) implies the convergence of \( \int_5^{\infty} \frac{dx}{e^{x^2}} \).
(3) The length is
\[
\int_0^{2\pi} \sqrt{r^2 + (dr/d\theta)^2} \, d\theta = \int_0^{2\pi} \sqrt{(1 - \cos \theta)^2 + \sin^2 \theta} \, d\theta = \int_0^{2\pi} \sqrt{2 - 2 \cos \theta} \, d\theta
\]
\[
= 2 \int_0^{2\pi} \sqrt{\frac{1 - \cos \theta}{2}} \, d\theta = 2 \int_0^{2\pi} \sqrt{\sin^2(\theta/2)} \, d\theta
\]
\[
= 2 \int_0^{2\pi} |\sin(\theta/2)| \, d\theta = 2 \int_0^{2\pi} \sin(\theta/2) \, d\theta = 8.
\]

(4) The area is \( \frac{1}{2} \int_0^{2\pi} (1 - \cos \theta)^2 \, d\theta = \frac{1}{2} \int_0^{2\pi} 1 - 2 \cos \theta + \cos^2 \theta \, d\theta = \frac{1}{2} (2\pi - 0 + \pi) = \frac{3\pi}{2}. \)

(5) The substitution \( u = 1 + \frac{9}{4} (x + 2) \) leads to

\[
\text{length} = \int_0^1 \sqrt{1 + \left( \frac{d}{dx} \right)^2} \, dx = \frac{8}{27} \left( \frac{11}{2} + \frac{9x}{4} \right)^{3/2} \bigg|_0^1.
\]

(6) A sphere with radius \( R \) is obtained by rotating the semicircle \( y = \sqrt{R^2 - x^2}, -R \leq x \leq R \) about the \( x \)-axis. In this case,

\[
\sqrt{1 + \left( \frac{dy}{dx} \right)^2} = \sqrt{1 + \left( \frac{-x}{\sqrt{R^2 - x^2}} \right)^2} = \sqrt{\frac{R^2}{R^2 - x^2}} = \frac{R}{\sqrt{R^2 - x^2}}.
\]

The surface area is \( 2\pi \int_{-R}^{R} \frac{R}{\sqrt{R^2 - x^2}} \, dx = 4\pi R^2. \)

(7) Multiplying \( r = \sin \theta \) by \( r \), we get \( r^2 = r \sin \theta \), which is \( x^2 + y^2 = y \). This is \( (x - 0)^2 + (y - 1/2)^2 = (1/2)^2 \). The center of the circle is \( (0, 1/2) \). The radius of the circle is \( 1/2 \).

(8) We can use \( x = \frac{6 \cos t}{3}, y = \frac{6 \sin t}{4}, 0 \leq t \leq 2\pi. \)

(9) The length is \( \int_1^2 \sqrt{4t^2 + 9t^4} \, dt = \int_1^2 t \sqrt{4 + 9t^2} \, dt. \) We can compute this integral using the substitution \( u = 4 + 9t^2. \)

(10)(a) \( \frac{dx}{dt} = \tan x \) leads to \( \int \frac{\cos x \, dx}{\sin x} = \int 1 \, dt, \ln |\sin x| = t + C, |\sin x| = e^t e^C = Be^t, \) where \( B = e^C \) is a constant. Now we get \( \sin x = \pm Be^t = Ae^t, \) where \( A \) is a constant. Substituting \( t = 0 \) and \( x = \pi/6, \) we get \( 1/2 = A. \) The initial value problem is solved by \( x = \sin^{-1}((1/2)e^t). \)

(10)(b) \( (4 + x^3)^{1/2} \frac{dy}{dx} = (xy)^2 \) leads to \( \int \frac{dy}{y^2} = \int \frac{x^2}{(4 + x^3)^{1/2}} \, dx, -\frac{1}{y} = \frac{2}{3} (4 + x^3)^{1/2} + C. \) Substituting \( x = 0 \) and \( y = -1, \) we get \( C = -\frac{1}{3}. \) The initial value problem is solved by \( y = \frac{3}{1 - 2\sqrt{4 + x^3}}. \)
(11) The temperature $T$ of the coffee is given by $T = 25 + Ce^{-kt}$. The initial condition $T(0) = 50$ implies $C = 25$. Now we know that the temperature $T$ of the coffee is given by $T = 25 + 25e^{-kt}$. Substituting $t = 1$, we get $30 = 25 + 25e^{-k}$, which implies $k = \ln 5$. The problem asks us to find $t$ such that $40 = 25 + 25e^{-kt}$. This last equation gives $e^{-kt} = 3/5$, hence $-kt = \ln (3/5)$. We solve for $t$ using $k = \ln 5$.

(12) We know $P(t) = \frac{20,000}{0.95} + Ce^{(0.95)t}$ and $P(30) = 0$. This gives us $C = -(400,000)e^{-3/2}$. The initial balance is $P(0) = 400,000 - (400,000)e^{-3/2}$.

(13) Since $f(x) = x^{-1}$, we get $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$, $f'''(x) = -6x^{-4}$. This implies $f(1) = 1$, $f'(1) = -1$, $f''(1) = 2$, $f'''(1) = -6$. Now we know

$$T_3(x) = 1 + (-1)(x - 1) + \frac{2(x - 1)^2}{2} + \frac{-6(x - 1)^3}{6} = 1 - (x - 1) + (x - 1)^2 - (x - 1)^3.$$  

We know $|f^{(4)}(u)| = 24|u^{-5}| \leq 24$ when $1 \leq u \leq 3/2$. This says that we can use $K = 24$. This implies

$$|f(3/2) - T_3(3/2)| \leq \frac{24|3/2 - 1|^4}{4!} = \frac{1}{16}.$$

(14)(a) The inequalities $-1 \leq \sin n \leq 1$ lead to $-1 \leq \frac{\sin n}{n} \leq 1$. Since $\lim_{n \to \infty} \frac{1}{n} = 0 = \lim_{n \to \infty} \frac{1}{n}$, the Squeeze Theorem implies $\lim_{n \to \infty} \frac{\sin n}{n} = 0$.

(14)(b) L’Hôpital’s Rule gives $\lim_{x \to \infty} \ln ((3x)^{1/x}) = \lim_{x \to \infty} \frac{\ln (3x)}{x} = \lim_{x \to \infty} \frac{1/x}{1} = 0$. Exponentiating this, we get $\lim_{x \to \infty} (3x)^{1/x} = e^0 = 1$. This implies $\lim_{n \to \infty} (3n)^{1/n} = 1$.

(14)(c) L’Hôpital’s Rule gives

$$\lim_{x \to \infty} \ln \left(\left(1 - \frac{5}{x}\right)^x\right) = \lim_{x \to \infty} x \ln \left(1 - \frac{5}{x}\right) = \lim_{x \to \infty} \frac{\ln (1 - \frac{5}{x})}{1/x} = \lim_{x \to \infty} \frac{\frac{5/x^2}{1-5/x}}{-1/x^2} = -5.$$  

Exponentiating this, we get $\lim_{x \to \infty} \left(1 - \frac{5}{x}\right)^x = e^{-5}$. This implies $\lim_{n \to \infty} \left(1 - \frac{5}{n}\right)^n = e^{-5}$.

(14)(d) Since $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and $\lim_{n \to \infty} 1/n = 0$, we conclude $\lim_{n \to \infty} \frac{\sin (1/n)}{1/n} = 1$. This is equivalent to $\lim_{n \to \infty} n \sin (1/n) = 1$.

(14)(e) L’Hôpital’s Rule gives $\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x} = \frac{1}{2}$. Since $\lim_{n \to \infty} 1/n = 0$, we conclude $\lim_{n \to \infty} \frac{1 - \cos (1/n)}{(1/n)^2} = \frac{1}{2}$. This is equivalent to $\lim_{n \to \infty} n^2 (1 - \cos (1/n)) = \frac{1}{2}$. 

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(15) Since $\left| \frac{1}{1000} \right| < 1$, the formula for the sum of a geometric series gives

$$5.273273273\ldots = 5 + \frac{273}{1000} + \frac{273}{(1000)^2} + \frac{273}{(1000)^3} + \cdots$$

$$= 5 + \frac{273}{1000} \left( 1 + \frac{1}{1000} + \left( \frac{1}{1000} \right)^2 + \left( \frac{1}{1000} \right)^3 + \cdots \right)$$

$$= 5 + \frac{273}{1000} \left( \frac{1}{1 - \frac{1}{1000}} \right) = 5 + \frac{273}{999} = \frac{5268}{999}.$$  

(16)(a) $\sum_{n=3}^{\infty} \frac{2^n}{3n+1} = \frac{2^3}{3^4} \left( 1 + \frac{2}{3} + \left( \frac{2}{3} \right)^2 + \left( \frac{2}{3} \right)^3 + \cdots \right) = \frac{2^3}{3^4} \cdot \frac{1}{1 - 2/3} = \frac{8}{27}$.

(16)(b) $\sum_{n=4}^{N} \frac{1}{n(n-1)} = \sum_{n=4}^{N} \left( \frac{1}{n-1} - \frac{1}{n} \right) = \frac{1}{3} - \frac{1}{N}$ because the other terms cancel out in pairs. Now

$$\sum_{n=4}^{\infty} \frac{1}{n(n-1)} = \lim_{N \to \infty} \sum_{n=4}^{N} \frac{1}{n(n-1)} = \lim_{N \to \infty} \left( \frac{1}{3} - \frac{1}{N} \right) = \frac{1}{3}.$$  

(17)(a) Since $\frac{1}{\sqrt{n}}$ is decreasing and approaches 0, the Leibniz Test tells us that $\sum_{n=5}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ converges.

(17)(b) $\int_{5}^{\infty} \frac{dx}{x(\ln x)} = \lim_{b \to \infty} \ln(\ln x)^5_{b} = \infty$, hence $\sum_{n=5}^{\infty} \frac{1}{n(\ln n)}$ diverges by the Integral Test.

(17)(c) $\int_{5}^{\infty} \frac{dx}{x(\ln x)^{3/2}} = \lim_{b \to \infty} \frac{2(\ln x)^{-1/2}}{5}^{b} = 2(\ln 5)^{-1/2} < \infty$, hence $\sum_{n=5}^{\infty} \frac{1}{n(\ln n)^{3/2}}$ converges by the Integral Test.

(17)(d) Since the answer to 14(c) is $\lim_{n \to \infty} \left( 1 - \frac{5}{n} \right)^n = e^{-5} \neq 0$, the Test For Divergence says that $\sum_{n=4}^{\infty} \left( 1 - \frac{5}{n} \right)^n$ diverges.

(17)(e) We do a limit comparison with the series $\sum_{n=4}^{\infty} \frac{1}{n}$. The answer to 14(d) says $\lim_{n \to \infty} \frac{\sin(1/n)}{1/n} = 1$. Since this limit is positive and finite, we conclude divergence of $\sum_{n=4}^{\infty} \sin(1/n)$ from the divergence of $\sum_{n=4}^{\infty} \frac{1}{n}$.  

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(17)(f) We do a limit comparison with the series \( \sum_{n=4}^{\infty} \frac{1}{n^2} \). The answer to 14(e) says
\[
\lim_{n \to \infty} \frac{1 - \cos(1/n)}{1/n^2} = \frac{1}{2}.
\]
Since this limit is positive and finite, we conclude convergence of
\( \sum_{n=4}^{\infty} (1 - \cos(1/n)) \) from the convergence of \( \sum_{n=4}^{\infty} \frac{1}{n^2} \).

(17)(g) Since \( |2/3| < 1 \), we know that \( \sum_{n=2}^{\infty} \frac{2n}{3n^2} = \sum_{n=2}^{\infty} \left( \frac{2}{3} \right)^n \) converges. The Comparison Test and \( 0 < \frac{2^n}{3^n + 1} < \frac{2^n}{3^n} \) allow us to conclude that \( \sum_{n=2}^{\infty} \frac{2^n}{3^n + 1} \) converges.

(17)(h) The Limit Comparison Test with \( \sum_{n=2}^{\infty} \frac{1}{n^2} \) is successful because
\[
\lim_{n \to \infty} \frac{n^2}{n^4 - n^3 - 4} = \lim_{n \to \infty} \frac{n^4}{n^4 - n^3 - 4} = 1,
\]
which is a positive and finite limit. Since the series \( \sum_{n=2}^{\infty} \frac{1}{n^2} \) converges, we conclude that the series \( \sum_{n=2}^{\infty} \frac{n^2}{n^4 - n^3 - 4} \) converges.

(18)(a) If we take the sum of only the first 10 terms of \( \sum_{n=1}^{\infty} \frac{1}{n^2} \), then the absolute value of the error is at most \( \int_{10}^{\infty} \frac{dx}{x^2} = \frac{1}{10} \).

(18)(b) If we take the sum of only the first 10 terms of \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \), then the absolute value of the error is at most the absolute value of the first omitted term, which is \( \frac{1}{11^2} \).

(19) An infinite series \( \sum a_n \) converges absolutely when \( \sum |a_n| \) converges. An infinite series \( \sum a_n \) converges conditionally when it converges, but does not converge absolutely. The series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \) converges conditionally.

(20) Let \( L \) denote \( \lim_{n \to \infty} a_n \). Taking the limit in the equation \( a_{n+1} = \sqrt{12 + a_n} \), we get
\[
L = \sqrt{12 + L}.
\]
The number \( L \) must be a solution of the equation \( L^2 = 12 + L \). This means that \( L \) must be either 4 or -3. The equation \( L = \sqrt{12 + L} \) excludes the possibility \( L = -3 \). We must have \( L = 4 \).