REVIEW PROBLEMS FOR MIDTERM TWO
MATH 151, SPRING 2008

1. Find the derivative of the following functions:

(a) $\tan(x^2 y) = x + y$. (b) $f(z) = \sin^{-1}\left(\frac{z}{1-z^2}\right)$. (c) $y = x^{\tan^{-1} x}$.

(d) $y = \frac{e^x \cos^{-1} x}{\ln x}$. (e) $y = 5^{x^2 - x}$. (f) $y = \sqrt{\frac{x(x+2)}{(2x+1)(3x-2)}}$.

2. Evaluate the following limits:

(a) $\lim_{x \to 0} \frac{x^3}{\sin x - x}$. (b) $\lim_{n \to \infty} \left(\frac{n+3}{n}\right)^n$. (c) $\lim_{x \to \infty} \frac{\ln(e^t + 1)}{t}$.

(d) $\lim_{x \to 0} \sqrt{x} \ln x$. (e) $\lim_{x \to 4} \left(\frac{1}{\sqrt{x} - 2} - \frac{4}{x - 4}\right)$. (f) $\lim_{x \to \infty} \frac{12x + 1}{\sqrt{4x^2 + 4x}}$.

3. Find the antiderivative of the following functions:

(a) $\int \sqrt{x}(x^2 - 1)dx$ (b) $\int \cos(3 - 4t)dt$ (c) $\int \frac{x^2 + 2x - 3}{x^4} dx$.

(d) $\int (4\theta + \sin 8\theta) d\theta$ (e) $\int \tan 3\theta \sec 3\theta d\theta$. (g) $\int (3x^{\frac{5}{2}} + 2x^{-1}) x^2 dx$

4. Find the maxima and minima for the given functions and the intervals.

(a) $y = \frac{1 - x}{x^2 + 3x}$ in $[1, 4]$. (b) $y = |3x^2 - 9|$ in $[-4, 5]$. (c) $y = x - \frac{4x}{x+1}$ in $[0, 3]$.

5. Sketch the graphs of the following functions (with increasing/decreasing, maxima/minima, concavity/points of inflections, and horizontal/vertical asymptotes):

(a) $y = 4 - 2x^2 + \frac{1}{6}x^4$. (b) $f(x) = \frac{1}{4 - x^2}$. (c) $f(x) = x^2 e^{-x}$.

6. Estimate $\sqrt{26} - \sqrt{25}$ using the linear approximation, and find an error using a calculator.

7. Using the Newton's method to estimate the solution of $e^x - 5x = 0$ up to three decimal places.

8. Find all points on the folium $x^3 + y^3 = 3xy$ at which the tangent line is horizontal.

9. Find the equations of the tangent lines of functions at given point or given slope of a tangent line:

(a) $ye^x + xe^y = 4$, $P = (4, 0)$. (b) $f(x) = x^3 - 3x^2 + x + 4$, slope $= 10$.

10. Suppose $f(1) = 5$, $f'(x) \geq 2$ for $x \geq 1$. Show that $f(8) \geq 24$.

11. Use the Mean value theorem to prove that $\sin x - \cos x = 3x$ has a solution, and use Rolle's theorem to show that this solution is unique.

12. The driver of an automobile applies the brakes at time $t = 0$ and comes to a halt after traveling 550 ft. Find the automobile's velocity at $t = 0$ assuming that the rate of deceleration was a constant -10 ft/s^2.

13. Suppose $f(g(x)) = e^{x^2}$, $g(1) = 2$ and $g'(1) = 4$. Find $f'(2)$.

14. A farmer is to build a rectangular fence of perimeter 1000 ft. Find the size of the rectangular that bounds the largest area.

15. Suppose f is a function defined for all x. Assume also that f' and f'' exist for all x. If

$$\begin{aligned}
f(0) &= -2, f(2) = 2, f(4) = 1 \\
f'(0) &= f'(2) = f'(4) = 0 \\
f''(-1) &= f''(1) = f''(3) = 0 \\
f'(x) &< 0 \text{ for } x < 0 \text{ and for } 2 < x < 4 \\
f'(x) &> 0 \text{ for } 0 < x < 2 \text{ and for } x < 4 \\
f''(x) &< 0 \text{ for } x < -1 \text{ and for } 1 < x < 3 \\
f''(x) &> 0 \text{ for } -1 < x < 1 \text{ and for } 3 < x
\end{aligned}$$

(a) Find all local minima and maxima inside the interval $-2 < x < 5$.
(b) Find all inflection points.
(c) Sketch the part of the graph of $y = f(x)$ that lies over the interval $-2 \leq x \leq 5$.

16. A rectangle with sides parallel to the x-axis ad the y-axis is inscribed in the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Of all such rectangles, find the dimensions of the one with maximum area.

17. In the right triangle $\triangle ABC$, the right angle is at C and the legs are $|AC| = 4$ and $|BC| = 12$. A rectangle is to placed inside the triangle, with one corner at C and the opposite corner on the hypotenuse. What are the dimensions and area of the largest rectangle that fits?

18. Suppose that $y(x)$ satisfies the equation $2xy - \ln y = 4$ and the condition $y(2) = 1$.

(a) Calculate $y'(2)$ and find the linear approximation $L(x)$ for $y(x)$ around $x = 2$.

(b) Obtain an equation relating dy and dx near $x = 2$.

(c) Using either a) or b), get an approximate value of $y(2.1)$.

(d) What is the concavity of the graph of $y(x)$ near $x = 2$?

(e) Is the linear approximation you found for $y(2.1)$ greater or smaller than the true value?

19. A spotlight on the ground shines on a wall 20 meters away. A woman 2 meters tall walks from the spotlight to the wall, at a sapped of 0.4 m/sec; her path is perpendicular to the wall.Let x be the distance from her feet to the spotlight and let h be the height of her shadow on the wall. Also let θ be the angle of elevation at the spotlight from the horizontal to the top of her head.

(a) Draw a sketch of the problem, and find a formula relating h and x.

(b) When the woman is 4 meters from the wall, find the height of her shadow and the rate of change of the height of her shadow.

(c) What is the rate of change of θ at this moment?