Partial solutions to the Math 151 review problems for Exam 2

These are not complete solutions. They are only intended as a way to check your work.

(1)(a) The derivative is \(-\frac{(\ln 10)10\arccos x}{\sqrt{1 - x^2}}\).

(1)(b) The derivative is \(\frac{3x^2 + 1}{(\ln 5)(x^3 + x + 1)\sqrt{1 - [\log_5(x^3 + x + 1)]^2}}\).

(1)(c) The derivative is \(\frac{1}{(1 + x^2)\tan^{-1} x}\).

(2)(a) The limit is \(-4\). The L'Hôpital Rule (used twice) is much more efficient than using long division to factor \((x - 1)^2\) out of the numerator and denominator.

(2)(b) The L'Hôpital Rule (used three times) gives \(1/3\). In Math 152 you will learn a more efficient way to find this limit.

(3)(a) \(\lim_{x \to 0^+} x^{1/10} \ln x = \lim_{x \to 0^+} \frac{\ln x}{x^{-1/10}} = \lim_{x \to 0^+} \frac{1/x}{-(1/10)x^{-11/10}} = \lim_{x \to 0^+} -10x^{1/10} = 0\).

(3)(b) \(\lim_{x \to \infty} \frac{\ln x}{x^{1/10}} = \lim_{x \to \infty} \frac{1/x}{(1/10)x^{-9/10}} = \lim_{x \to \infty} 10x^{-1/10} = 0\).

(4) The horizontal asymptotes are \(y = \frac{1}{\sqrt{7}}\) and \(y = -\frac{1}{\sqrt{7}}\).

(5)(a) We get \(f'(x) = -\frac{8x}{(x^2 - 4)^2}\), \(f''(x) = \frac{24x^2 + 32}{(x^2 - 4)^3}\). The function is increasing on \((-\infty, -2)\) and \((-2, 0)\). The function is decreasing on \((0, 2)\) and \((2, \infty)\). The function is concave up on \((-\infty, -2)\) and \((2, \infty)\). The function is concave down on \((-2, 2)\). There is a local maximum at \(x = 0\). There are no local minima. There are no inflection points. The horizontal asymptote is \(y = 1\). The vertical asymptotes are \(x = \pm 2\).

(5)(b) We get \(g'(x) = -\frac{x^2 + 4}{(x^2 - 4)^2}\), \(g''(x) = \frac{x(2x^2 + 24)}{(x^2 - 4)^3}\). The function is not increasing on any interval. The function is decreasing on \((-\infty, -2)\), \((-2, 2)\), \((2, \infty)\). The function is concave up on \((-2, 0)\) and \((2, \infty)\). The function is concave down on \((-\infty, -2)\), \((0, 2)\). There are no local extrema. There is an inflection point at \(x = 0\). The horizontal asymptote is \(y = 0\). The vertical asymptotes are \(x = \pm 2\).

(6) The function is increasing on \((-\infty, -1)\), \((-2/\sqrt{5}, 2/\sqrt{5})\), \((1, \infty)\). The function is decreasing on \((-1, -2/\sqrt{5})\) and \((2/\sqrt{5}, 1)\). There are local maxima at \(x = -1\) and \(x = 2/\sqrt{5}\). There are local minima at \(x = -2/\sqrt{5}\) and \(x = 1\). The function is concave up
on \((-3/\sqrt{10}, 0)\) and \((3/\sqrt{10}, \infty)\). The function is concave down on \((-\infty, -3/\sqrt{10})\) and \((0, 3/\sqrt{10})\). The inflection points are at \(x = 0\) and \(x = \pm 3/\sqrt{10}\).

(7) We get \(f'(x) = -\frac{x}{(x^2 + 1)^{3/2}}\) and \(f''(x) = \frac{2x^2 - 1}{(x^2 + 1)^{5/2}}\). The function is concave up on \((-\infty, -1/\sqrt{2})\) and \((1/\sqrt{2}, \infty)\). The function is concave down on \((-1/\sqrt{2}, 1/\sqrt{2})\). The inflection points are at \(x = \pm 1/\sqrt{2}\).

(8) The local maxima (which are also absolute maxima) occur at \(-\pi/6 + 2n\pi\). The local minima (which are also absolute minima) occur at \(5\pi/6 + 2n\pi\). The number \(n\) represents an arbitrary integer.

(9) Use \(x^2 = |x|^2 = |x| \cdot |x| = |x|\sqrt{x^2}\) to obtain the formula as it is usually written.

(10) The absolute minimum is at \(x = -\sqrt{2}\). The absolute maximum is at \(x = -1\).

(11) The derivative is \((\ln x)^2 (\ln(\ln x) + 1/\ln x)\).

(12) The slope of the tangent is \(-50/111\). In order to appreciate the usefulness of implicit differentiation, try the following cumbersome alternative method: Solve for \(x\), find the derivative \(dx/dy\), take the reciprocal of this derivative evaluated at \(y = 2\).

(13) The linearization is \(L(x) = 2 + x/27\). The linearization is greater than \(f(x)\) when \(x \neq 27\).

(14) The volume increases by approximately 0.003 percent. The surface area increases by approximately 0.002 percent.

(15) If \(R\) is the radius, \(V\) is the volume and \(A\) is the area, then the formulas \(V = (4/3)\pi R^3\) and \(A = 4\pi R^2\) lead to \(\frac{12}{5} = \frac{dV/dt}{dA/dt} = \frac{4\pi R^2(dR/dt)}{8\pi R(dR/dt)} = \frac{R}{2}\). We are giving you all of the details of the answer because forgetting the \((dR/dt)\) in the Chain Rule would lead to the same correct answer \(R = 24/5\) inches. Getting the right final answer does not guarantee that the method is correct.

(16) We can use thin rectangles of width \(1/n\) with right upper vertex on the curve \(y = x^2\). The total area of these rectangles is

\[
\sum_{j=1}^{n} \frac{1}{n} \left( \frac{j}{n} \right)^2 = \frac{1}{n^3} \sum_{j=1}^{n} j^2 = \frac{1}{n^3} \left( \frac{n(n+1)(2n+1)}{6} \right) = \frac{1}{6} \cdot \frac{n+1}{n} \cdot \frac{2n+1}{n}.
\]

This approaches the correct area \(1/3\) as \(n\) approaches infinity.

(17) Use \(f(x) = x^2 - 7\) and a positive initial guess. If you use a negative initial guess, then you will approximate \(-\sqrt{7}\).
(18) \( f(x) = \frac{x^5}{20} - \frac{x^3}{6} + Cx + D \) for any \( C \) and \( D \).

(19) We get
\[
 f'(x) = (2/3)x^{-1/3}(1 - x^2)(1 - 7x^2)
\]
and
\[
 f''(x) = (2/9)x^{-4/3}(77x^4 - 40x^2 - 1).
\]
The function is increasing on \((-1, -1/\sqrt{7}), (0, 1/\sqrt{7}), (1, \infty)\). The function is decreasing on \((-\infty, -1), (-1/\sqrt{7}, 0), (1/\sqrt{7}, 1)\). There are local maxima at \( x = \pm 1/\sqrt{7} \). There are local minima (which are absolute minima) at \( \pm 1 \) and 0. There are inflection points at
\[
 x = \pm \sqrt{\frac{20 + \sqrt{477}}{77}}.
\]
The function is concave up on
\[
 \left( -\infty, -\sqrt{\frac{20 + \sqrt{477}}{77}} \right), \quad \left( \sqrt{\frac{20 + \sqrt{477}}{77}}, \infty \right).
\]
The function is concave down on
\[
 \left( -\sqrt{\frac{20 + \sqrt{477}}{77}}, 0 \right), \quad \left( 0, \sqrt{\frac{20 + \sqrt{477}}{77}} \right).
\]

(20) We get \( f'(x) = \frac{1 - x^2}{(1 + x^2)^2} \) and \( f''(x) = \frac{x(2x^2 - 6)}{(1 + x^2)^3} \). The function is increasing on \((-1, 1)\). The function is decreasing on \((-\infty, -1)\) and \((1, \infty)\). The function is concave up on \((-\sqrt{3}, 0)\) and \((\sqrt{3}, \infty)\). The function is concave down on \((-\infty, -\sqrt{3})\) and \((0, \sqrt{3})\). There is an absolute maximum at \( x = 1 \). There is an absolute minimum at \( x = -1 \). There are inflection points at \( x = 0 \) and \( x = \pm \sqrt{3} \).

(21) The volume is maximized when \( x = \sqrt{35/3} \) feet and \( y = (2/7)\sqrt{35/3} \) feet.

(22) The surface area is minimized when the height of the cone is \((600/\pi)^{1/3}\) inches. The surface area is the area of the “pacman” obtained when we make a straight line cut in the cone from the vertex to a point on the base of the cone, and then unravel the cone so that it is completely flat.

(23) The answer should have \( H = 2R \), where \( H \) is the height of the cylinder, and \( R \) is the radius of the top (and bottom) circle.

(24) The inequality \( f(7) < -1 \) is a consequence of the Mean Value Theorem.