Partial Solutions to Review Problems for Exam 1

(1) The first inequality gives us the set which is the union of the intervals \((-\infty, 1)\) and \((3, \infty)\). The second inequality gives us the set which is just the interval \([2, 4]\). The intersection of these two sets gives us the answer to the given question. This answer is the interval \((3, 4)\).

(2)(a) We have to show that the inequalities \(x_1 \leq x_2 \leq -5\) imply \(f(x_1) \leq f(x_2)\). The assumption \(x_1 \leq x_2 \leq -5\) gives us \(5 \leq -x_2 \leq -x_1\). Since \(f(x)\) is odd and increasing on \([5, \infty)\), we conclude \(-f(x_2) \leq f(-x_1) = -f(x_1)\). Therefore, \(f(x_1) \leq f(x_2)\).

(2)(b) We have to show that the inequalities \(x_1 \leq x_2 \leq -5\) imply \(f(x_1) \leq f(x_2)\). The assumption \(x_1 \leq x_2 \leq -5\) gives us \(5 \leq -x_2 \leq -x_1\). Since \(f(x)\) is even and increasing on \([5, \infty)\), we conclude \(f(x_2) = f(-x_2) \leq f(-x_1) = f(x_1)\). Therefore, \(f(x_1) \geq f(x_2)\).

(3) To begin with, \(2x^2 - 8x - 10 = 2(x^2 - 4x - 5) = 2((x - 2)^2 - 9)\). The minimum of \(2x^2 - 8x - 10\) occurs at the \(x\) which minimizes \((x - 2)^2 - 9\). This particular \(x\) is the \(x\) that makes \((x - 2)^2\) equal to 0. That \(x\) is 2. In other words, the minimum of \(2x^2 - 8x - 10\) occurs at \(x = 2\), and that minimum is \(-18\). The equation \(0 = 2x^2 - 8x - 10 = 2((x - 2)^2 - 9)\) leads to \(x - 2 = \pm 3\), \(x = -1\) and \(x = 5\).

(4) There are many choices for \(f(x)\) and \(g(x)\). The choices \(f(x) = x^2\) and \(g(x) = x + 1\) will work. Indeed, \((f \circ g)(x) = f(g(x)) = (x + 1)^2\) does not equal \((g \circ f)(x) = g(f(x)) = x^2 + 1\) for general values of \(x\).

(5) The given equation is equivalent to \(2\sin^2 x = 1 + \cos^2 x - \sin^2 x = 1 + 1 - \sin^2 x - \sin^2 x\), which is equivalent to \(\sin^2 x = 1/2\). We need to find all \(x\) in \([0, 2\pi]\) with the property \(\sin x = \pm \sqrt{1/2} = \pm \sqrt{2}/2\). These \(x\) are \(\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\).

(6) The periodicity of \(\sin\) gives \(\sin^{-1}(\sin(9\pi/4)) = \sin^{-1}(\sin(\pi/4)) = \pi/4\). The fact \(\cos(\sin^{-1} x) = \sqrt{1 - x^2}\) (see page 39 in the textbook) leads to \(\sec(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}\).

If we modify the argument in the text that led to the second proof of \(\cos(\sin^{-1} x) = \sqrt{1 - x^2}\) on page 39, then we get the following: If \(\theta = \tan^{-1} x\) then \(\sec(\tan^{-1} x) = \sec(\theta) = \sqrt{1 + \tan^2 \theta} = \sqrt{1 + x^2}\), where we took the positive square root because \(\theta = \tan^{-1} x\) lies in \((-\pi/2, \pi/2)\) and \(\sec\theta\) is positive in this interval. Now we use the fact \(\sec(\tan^{-1} x) = \sqrt{1 + x^2}\) to conclude \(\cos(\tan^{-1} x) = \frac{1}{\sqrt{1 + x^2}}\).

(7) The given identity is equivalent to \(\ln \left(\frac{x^2 + 7}{x^2 + 1}\right) = \ln(2^2)\). Since the \(\ln\) function is one-to-one, we conclude \(\frac{x^2 + 7}{x^2 + 1} = 2^2 = 4\). Consequently, \(x^2 = 1\) and \(x = \pm 1\).

(8) The average velocity is \(\frac{1}{3-1} \left(\frac{3}{1+3x} - \frac{1}{1+x^2}\right) = -0.1\) feet per second.

(9) We will use the letters \((a) - (m)\) to label these 13 limit problems.
(a) If we take the reciprocal of \( \lim_{u \to 0} \frac{\sin u}{u} = 1 \) then the limit laws give us \( \lim_{u \to 0} \frac{u}{\sin u} = 1 \). If \( u = 7x \) then \( x \to 0 \) is the same as \( u \to 0 \). Consequently,

\[
\lim_{x \to 0} \frac{x}{\sin(7x)} = \frac{1}{7} \lim_{x \to 0} \frac{7x}{\sin(7x)} = \frac{1}{7} \lim_{u \to 0} \frac{u}{\sin u} = \frac{1}{7} \cdot 1 = \frac{1}{7}.
\]

(b) Imitating the method in (a), we set \( u = 5x, v = 7x \) and note that \( x \to 0 \) is the same as \( u \to 0 \) and the same as \( v \to 0 \). Consequently,

\[
\lim_{x \to 0} \frac{\sin(5x)}{\sin(7x)} = \lim_{x \to 0} \frac{5}{7} \cdot \frac{\sin(5x)}{5x} \cdot \frac{7x}{\sin(7x)} = \frac{5}{7} \lim_{u \to 0} \frac{\sin u}{u} \cdot \lim_{v \to 0} \frac{v}{\sin v} = \frac{5}{7} \cdot 1 \cdot 1 = \frac{5}{7}.
\]

(c) \( \lim_{x \to 0} \frac{x}{\tan x} = \lim_{x \to 0} \frac{x}{\sin x} \cdot \cos x = 1 \cdot 1 = 1 \).

(d) We are dealing with nonzero values of \( x \). The inequality \( |\cos(x^{-3})| \leq 1 \) leads to \( |x| \cdot |\cos(x^{-3})| \leq |x| \cdot 1 \), which is \( |x \cos(x^{-3})| \leq |x| \). This last inequality can be rewritten

\[-|x| \leq x \cos(x^{-3}) \leq |x|.
\]

Since \( \lim_{x \to 0} -|x| = 0 = \lim_{x \to 0} |x| \), the Squeeze Theorem implies \( \lim_{x \to 0} x \cos(x^{-3}) = 0 \).

(e) When \( x \) approaches 5 from the right, we always have \( x > 5 \), which is the same as \( x - 5 > 0 \). The inequality \( x - 5 > 0 \) allows us to write \( |x - 5| = x - 5 \). Now we know

\[
\lim_{x \to 5^+} \frac{x - 5}{|x - 5|} = \lim_{x \to 5^+} \frac{x - 5}{x - 5} = \lim_{x \to 5^+} 1 = 1.
\]

(f) When \( x \) approaches 5 from the left, we always have \( x < 5 \), which is the same as \( x - 5 < 0 \). The inequality \( x - 5 < 0 \) allows us to write \( |x - 5| = -(x - 5) \). Now we know

\[
\lim_{x \to 5^-} \frac{x - 5}{|x - 5|} = \lim_{x \to 5^-} \frac{x - 5}{-(x - 5)} = \lim_{x \to 5^-} -1 = -1.
\]

(g) The general fact \( \lim_{x \to a^+} \frac{1}{x - a} = \infty \) leads to

\[
\lim_{x \to 3^+} \frac{x^2 - 20}{x^2 - 9} = \lim_{x \to 3^+} \left( \frac{x^2 - 20}{x^2 - 9} \cdot \frac{1}{x + 3} \right) = \lim_{x \to 3^+} \frac{x^2 - 20}{x^3} \cdot \lim_{x \to 3^+} \frac{1}{x - 3} = \frac{11}{6} \infty = -\infty.
\]

(h) The general fact \( \lim_{x \to a^-} \frac{1}{x - a} = -\infty \) leads to

\[
\lim_{x \to 3^-} \frac{x^2 - 20}{x^2 - 9} = \lim_{x \to 3^-} \left( \frac{x^2 - 20}{x^2 - 9} \cdot \frac{1}{x + 3} \right) = \lim_{x \to 3^-} \frac{x^2 - 20}{x^3} \cdot \lim_{x \to 3^-} \frac{1}{x - 3} = \frac{-11}{6} (-\infty) = \infty.
\]

(i) \( \lim_{x \to 2} \frac{x^2 + x - 6}{x^2 + 2x - 8} = \lim_{x \to 2} \frac{(x - 2)(x + 3)}{(x - 2)(x + 4)} = \lim_{x \to 2} \frac{x + 3}{x + 4} = \frac{5}{6} \).
(i) \[ \lim_{x \to 2} \frac{x^3 - 2x^2 + x - 2}{x^3 - x^2 - x - 2} = \lim_{x \to 2} \frac{(x - 2)(x^2 + 1)}{(x - 2)(x^2 + x + 1)} = \lim_{x \to 2} \frac{x^2 + 1}{x^2 + x + 1} = \frac{5}{7}. \]

(k) A correct rationalization starts with
\[ \lim_{x \to 3} \frac{4 - \sqrt{5x + 1}}{5 - \sqrt{8x + 1}} = \lim_{x \to 3} \frac{(4 - \sqrt{5x + 1})(4 + \sqrt{5x + 1})(5 + \sqrt{8x + 1})}{5 - \sqrt{8x + 1})(4 + \sqrt{5x + 1})(5 + \sqrt{8x + 1})}. \]

Using the basic identity \((a - b)(a + b) = a^2 - b^2\) twice, we get
\[ \lim_{x \to 3} \frac{4 - \sqrt{5x + 1}}{5 - \sqrt{8x + 1}} = \lim_{x \to 3} \frac{(16 - (5x + 1))(5 + \sqrt{8x + 1})}{(25 - (8x + 1))(4 + \sqrt{5x + 1})} = \lim_{x \to 3} \frac{5(3 - x)(5 + \sqrt{8x + 1})}{8(3 - x)(4 + \sqrt{5x + 1})} = \frac{5(3 - x)}{8(3 - x)} = \frac{5}{8}.
\]

(l) \[ \lim_{x \to 3} \frac{4 - \sqrt{5x + 1}}{6 - 2x} = \lim_{x \to 3} \frac{(4 - \sqrt{5x + 1})(4 + \sqrt{5x + 1})}{(6 - 2x)(4 + \sqrt{5x + 1})} = \lim_{x \to 3} \frac{16 - (5x + 1)}{6 - 2x} = \frac{5}{16}. \]

(m) \[ \lim_{x \to 0} \frac{1 - \sec x}{x^2} = \lim_{x \to 0} \frac{(1 - \sec x)(1 + \sec x)}{x^2(1 + \sec x)} = \lim_{x \to 0} \frac{1}{x^2(1 + \sec x)} = \lim_{x \to 0} \frac{-\tan^2 x}{x^2} = \lim_{x \to 0} \frac{-2x}{x^2} = -2. \]

(10) We must have \(2c = f(1) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} (bx + c) = b + c = b + 2\) gives \(b = 2\). Finally, we get \(a = -2\) from \(a + 2 = a + b = f(-1) = \lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (bx + c) = -b + c = -2 + 2 = 0\). One should check that the identities \(\lim_{x \to 1} f(x) = f(-1)\) and \(\lim_{x \to 1} f(x) = f(-1)\) do hold for this choice of \(a, b, c\).

(11) Let us use the auxiliary function \(f(x) = x - \cos x\). We verify that \(f(x)\) is continuous on \([0, \pi/2]\) and that we have \(f(0) < 0 < f(\pi/2)\). The Intermediate Value Theorem tells us that \(f(c) = 0\) is true for some \(c\) in the interval \((0, \pi/2)\). This \(c\) is a solution of \(x = \cos x\).

(12) For each \(\varepsilon > 0\) we can choose \(\delta = \varepsilon/3 > 0\) and verify that the condition \(0 < |x - 2| < \delta\) implies \(|(3x + 4) - 10| = |3x - 6| = 3|x - 2| < 3\delta = \varepsilon\).

(13) Using \(f(x) = \frac{1}{x^2}\), \(f(x + h) = \frac{1}{(x + h)^2}\) we get
\[ \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{(x + h)^2} - \frac{1}{x^2} = \lim_{h \to 0} \frac{x^2 - (x + h)^2}{h(x + h)^2x^2} = \frac{-2x - h^2}{h(x + h)^2x^2} = \frac{-2x}{x^2x^2} = -2x^{-3}. \]
(14) If \( t \) is the time when the outfielder caught the ball, then we must have \(-16t^2 + 12t + 4 = 6\). The two solutions are \( t = 1/4 \) seconds (when the ball is on its way up) and \( t = 1/2 \) seconds (when the ball is on its way down). I may have seen once a pitcher who caught a batted ball on its way up, but an outfielder expects to catch a fly ball on its way down. If the catch was routine then the answer to the first question is \( t = 1/2 \) seconds. The maximum height occurs when the derivative of the height function is zero. This corresponds to \( t = 3/8 \) seconds and the maximum height \(-16(3/8)^2 + 12(3/8) + 4 \) feet.

(15) We see \( \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x + 3) = 2 + 3 = 5 = 3 + 2 = \lim_{x \to 1^+} (3x + 2) = \lim_{x \to 1^+} f(x) \). This implies \( \lim_{x \to 1} f(x) = 5 \). We also see \( f(1) = 3 + 2 = 5 \). Now we have \( \lim_{x \to 1^-} f(x) = f(1) \), which proves continuity at 1, the only point where continuity could possibly be in doubt. Now we will show that \( f'(1) \) does not exist. If \( f'(1) \) did exist, then \( \lim_{h \to 0^-} \frac{f(1 + h) - f(1)}{h} \) would exist, and this would imply

\[
\lim_{h \to 0^-} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0^+} \frac{f(1 + h) - f(1)}{h},
\]

which is the same as

\[
\lim_{h \to 0^-} \frac{f(1 + h) - 5}{h} = \lim_{h \to 0^+} \frac{f(1 + h) - 5}{h}.
\]

We get a contradiction because the computations

\[
\lim_{h \to 0^-} \frac{(2(1 + h) + 3) - 5}{h} = \lim_{h \to 0^-} \frac{2h}{h} = 2
\]

and

\[
\lim_{h \to 0^+} \frac{(3(1 + h) + 2) - 5}{h} = \lim_{h \to 0^+} \frac{3h}{h} = 3
\]

tell us that the earlier assertion

\[
\lim_{h \to 0^-} \frac{f(1 + h) - 5}{h} = \lim_{h \to 0^+} \frac{f(1 + h) - 5}{h}
\]

implies the conclusion \( 2 = 3 \), which is false. A quick way to visualize the nondifferentiability of this \( f(x) \) is to note the geometry at 1 on the \( x \)-axis. The slope from the right is 3, but the slope from the left is 2. This produces a corner at 1.

(16) Since \( f(x) \) is differentiable at 2, we conclude that \( f(x) \) is continuous at 2. This implies \( 5 = \lim_{x \to 2^-} (2x + 1) = \lim_{x \to 2^-} f(x) = f(2) = 4a + b \). This gives us the equation \( 5 = 4a + b \), which relates \( a \) and \( b \). As in problem (15), we can visualize what is going on at the point 2 on the \( x \)-axis. The slope from the right is \( 2ax \) at \( x = 2 \), which is \( 4a \). The slope from the left is 2. To avoid a corner, we need \( 4a = 2 \). What follows is a more rigorous analysis that also leads to \( 4a = 2 \). The differentiability of \( f(x) \) at 2 implies

\[
\lim_{h \to 0^-} \frac{f(2 + h) - f(2)}{h} = \lim_{h \to 0^+} \frac{f(2 + h) - f(2)}{h}.
\]
The above, the fact \( f(2) = 4a + b = 5 \) and the definition of \( f \) give
\[
\lim_{h \to 0^-} \frac{(2(2 + h) + 1) - 5}{h} = \lim_{h \to 0^+} \frac{(a(2 + h)^2 + b) - 5}{h},
\]
which simplifies to
\[
\lim_{h \to 0^-} \frac{2h}{h} = \lim_{h \to 0^+} \frac{4ah + ah^2}{h}.
\]
We used \( 4a + b = 5 \) in this simplification. Further simplification leads to \( \lim_{h \to 0^-} 2 = \lim_{h \to 0^+} (4a + ah) \), which is \( 2 = 4a \), as promised earlier. Solving the system \( 5 = 4a + b \) and \( 2 = 4a \), we get \( b = 3 \) and \( a = 1/2 \).

(17) The Product Rule, Quotient Rule, Chain Rule and other simpler rules give
\[
\frac{d}{dx} [(x^3 + x)^5(1 + \cos x)^9] = 5(x^3 + x)^4(3x^2 + 1)(1 + \cos x)^9 + (x^3 + x)^59(1 + \cos x)^8(- \sin x)
\]
\[
\frac{d}{dx} \left[ \tan x \right] = \frac{(\sec^2 x)(1 + e^{4x}) - 4e^{4x} \tan x}{(1 + e^{4x})^2}
\]
\[
\frac{d}{dx} \left[ \sin (\sqrt{x^4 + x^2 + 3}) \right] = \frac{1}{2}(x^2 + x^2 + 3)^{-1/2}(4x^3 + 2x) \cos \left( \sqrt{x^4 + x^2 + 3} \right)
\]
\[
\frac{d}{dx} \left[ \sec(x + \sqrt{x}) \right] = (e^x + (1/2)x^{-1/2}) \sec(e^x + \sqrt{x}) \tan(e^x + \sqrt{x})
\]

(18) If \( f(x) = (3 + x^{-3})^5 \) then \( f''(x) = 60x^{-5}(3 + x^{-3})^4 + 180x^{-8}(3 + x^{-3})^{-3} \).
If \( g(x) = \tan(7x) \) then \( g''(x) = 98 \sec^2(7x) \tan(7x) \).
If \( h(x) = (e^x + \cos x)^{-1/2} \) then
\[
h''(x) = \frac{3}{4}(e^x + \cos x)^{-5/2}(e^x - \sin x)^2 - \frac{1}{2}(e^x + \cos x)^{-3/2}(e^x - \cos x)
\]
If \( k(x) = e^{x^2+4x+3} \) then \( k''(x) = ((2x + 4)^2 + 2)e^{x^2+4x+3} \).

(19) If \( f(x) = \cos(2x) \) then \( f'(x) = -2 \sin(2x) \), \( f''(x) = -4 \cos(2x) \), \( f^{(3)}(x) = 8 \sin(2x) \), \( f^{(4)}(x) = 16 \cos(2x) \).

(20) The identity \( 0 = f''(x) = (4x^2 - 2)e^{-x^2} \) gives the solutions \( x = \pm 2^{-1/2} \). The identity \( 0 = g''(x) = \frac{6x^2 - 2}{(1+x^2)^3} \) gives the solutions \( x = \pm 3^{-1/2} \).