3.1. **Piecewise linear approximation.** Consider in more detail the case of continuous, piecewise linear approximation. Define the continuous, piecewise linear interpolant of a function f as the continuous, piecewise linear function $Q_1(x)$ satisfying $Q_1(x_j) = f(x_j)$, $j = 0, \dots n$. Using the Lagrange form of the interpolating polynomial, can write this as:

$$Q_1(x) = \frac{x - x_i}{x_{i-1} - x_i} f(x_{i-1}) + \frac{x - x_{i-1}}{x_i - x_{i-1}} f(x_i), \quad x \in [x_{i-1}, x_i].$$

A useful basis for the space of continuous, piecewise linear functions is the set $\{\psi_i\}_{i=0}^n$, where

$$\begin{aligned} \psi_i(x) &= 0, \quad x \notin [x_{i-1}, x_{i+1}], \\ &= (x - x_{i-1})/(x_i - x_{i-1}), \quad x \in [x_{i-1}, x_i], \\ &= (x_{i+1} - x)/(x_{i+1} - x_i), \quad x \in [x_i, x_{i+1}]. \end{aligned}$$

The basis function $\psi_i(x)$ is called a hat function. Note that $\psi_i(x_j) = 0$ for $i \neq j$ and $= 1$ for $i = j$. Hence, we can write

$$Q_1(x) = \sum_{i=0}^{n} \psi_i(x) Q_1(x_i) = \sum_{i=0}^{n} \psi_i(x) f(x_i).$$

In this form, the degrees of freedom for $Q_1(x)$ are the values $Q_1(x_i)$, and thus the solution of the interpolation problem is simple.

When the points x_j are equally spaced, we get a simplification. Let

$$\begin{aligned} \phi(x) &= 0, \quad x \geq 1, \text{ and } x \leq -1, \\ &= 1 - x, \quad 0 \leq x \leq 1, \\ &= 1 + x, \quad -1 \leq x \leq 0. \end{aligned}$$

Then $\psi_i(x) = \phi([x - x_i]/h)$, where $h = x_{i+1} - x_i$.

How good an approximation is the continuous piecewise linear interpolant? On each subinterval, $Q(x)$ is just the linear interpolating polynomial. Hence, using the error formula, we have for $x \in [x_{i-1}, x_i]$,

$$|f(x) - Q_1(x)| \leq M_{2,i}(x_i - x_{i-1})^2/8,$$

where $M_{2,i} = \max_{x_{i-1} \leq \xi \leq x_i} |f''(\xi)|$. Hence,

$$|f(x) - Q_1(x)| \leq M_2 \max_{i=1,n} (x_i - x_{i-1})^2/8 \leq M_2 h^2/8,$$

where $M_2 = \max_{i=1,n} M_{2,i} = \max_{x_0 \leq \xi \leq x_n} |f''(\xi)|$ and $h = \max_{i=1,n} |x_i - x_{i-1}|$.

Hence, if $f \in C^2[a, b]$, then taking more subintervals and letting the subinterval size approach zero, we can make the error as small as desired.

3.2. **Piecewise cubic Hermite approximation.** Next consider the piecewise cubic Hermite interpolant, i.e., a C^1 piecewise cubic $Q_{3,1}(x)$ satisfying

$$Q_{3,1}(x_j) = f(x_j), \quad Q'_{3,1}(x_j) = f'(x_j), \quad j = 0, \dots n.$$

On the subinterval $[x_{i-1}, x_i]$, $Q_{3,1}$ is just the cubic polynomial satisfying:

$$Q_{3,1}(x_{i-1}) = f(x_{i-1}), \quad Q_{3,1}(x_i) = f(x_i), \quad Q'_{3,1}(x_{i-1}) = f'(x_{i-1}), \quad Q'_{3,1}(x_i) = f'(x_i),$$

and so for $x \in [x_{i-1}, x_i]$,

$$Q_{3,1}(x) = f(x_{i-1}) + f'(x_{i-1})(x - x_{i-1}) + f[x_{i-1}, x_{i-1}, x_i](x - x_{i-1})^2 \\ + f[x_{i-1}, x_{i-1}, x_i, x_i](x - x_{i-1})^2(x - x_i).$$

On the interval $[x_{i-1}, x_i]$, the error

$$\begin{aligned} |f(x) - Q_{3,1}(x)| &= \frac{|f^{(4)}(\xi_i)|}{4!}(x - x_{i-1})^2(x - x_i)^2 \\ &\le \frac{M_{4,i}}{4!}\frac{(x_i - x_{i-1})^4}{16} \le M_4 h^4/384, \end{aligned}$$

where $M_{4,i} = \max_{x_{i-1} \le \xi \le x_i} |f^{(4)}(\xi)|$, $M_4 = \max_{i=1,n} M_{4,i} = \max_{x_0 \le \xi \le x_n} |f^{(4)}(\xi)|$, and $h = \max_{i=1,n} |x_i - x_{i-1}|$.

It is also useful to have a representation of $Q_{3,1}$ that uses the degrees of freedom $Q_{3,1}(x_j)$ and $Q'_{3,1}(x_j)$, $j = 0, 1, \dots, n$. Since the dimension of the space of piecewise C^1 cubics is $2n + 2$, we need to find basis functions $\phi_i(x)$, $\psi_i(x)$, $i = 0, 1, \dots, n$ that are C^1 piecewise cubics so that

$$Q_{3,1}(x) = \sum_{i=0}^{n} \phi_i(x) Q_{3,1}(x_i) + \sum_{i=0}^{n} \psi_i(x) Q'_{3,1}(x_i).$$

We do this by finding $\phi_i(x)$ and $\psi_i(x)$ satisfying:

$$\begin{aligned} &\phi_i(x_i) = 1, \ \phi_i(x_j) = 0, \ j \ne i, \ \phi'_i(x_j) = 0, \text{ for all } j, \\ &\psi_i(x_j) = 0, \text{ for all } j, \ \psi'_i(x_i) = 1, \ \psi'_i(x_j) = 0, \ j \ne i. \end{aligned}$$

We see that this implies that $\phi_i(x) = 0$ and $\psi_i(x) = 0$ for $x \le x_{i-1}$ and $x \ge x_{i+1}$. On the subintervals (x_{i-1}, x_i) and (x_{i1}, x_{i+1}), we have:

$$\begin{aligned} \phi_i(x) &= \left(\frac{x - x_{i-1}}{h_i}\right)^2 \left(1 - 2\frac{x - x_i}{h_i}\right), \quad x_{i-1} < x < x_i, \\ \phi_i(x) &= \left(\frac{x_{i+1} - x}{h_{i+1}}\right)^2 \left(1 + 2\frac{x - x_i}{h_{i+1}}\right), \quad x_i < x < x_{x+1}, \\ \psi_i(x) &= (x - x_i)\left(\frac{x - x_{i-1}}{h_i}\right)^2, \quad x_{i-1} < x < x_i, \\ \psi_i(x) &= (x - x_i)\left(\frac{x_{i+1} - x}{h_{i+1}}\right)^2, \quad x_i < x < x_{i+1}, \end{aligned}$$

We can reduce the work in finding these functions by noting that since $\phi_i(x_{i-1})$ and $\phi'_i(x_{i-1}) = 0$, $(x - x_{i-1})^2$ must be a factor of $\phi_i(x)$ on the subinterval (x_{i-1}, x_i) and by the defining conditions, $\psi_i(x)$ must be a multiple of $(x - x_i)(x - x_{i-1})^2$ on that subinterval. Similar formulas hold on the subinterval (x_i, x_{i+1}).

When the mesh points are equally spaced at a distance h apart, we can define basis functions for the space of piecewise cubic Hermite functions in the following simple way. We first define the functions

$$\phi(x) = (x+1)^2(1-2x), \quad -1 < x < 0, \qquad \phi(x) = (1-x)^2(1+2x), \quad 0 < x < 1,$$
$$\psi(x) = x(x+1)^2, \quad -1 < x < 0, \qquad \psi(x) = x(1-x)^2, \qquad 0 < x < 1,$$

with $\phi(x) = 0$ and $\psi(x) = 0$ for $x \le -1$ and $x \ge 1$. Then, the $2n+2$ basis functions are given by

$$\phi_i(x) = \phi([x - x_i]/h), \qquad \psi_i(x) = h\psi([x - x_i]/h), \qquad i = 0, \dots, n.$$

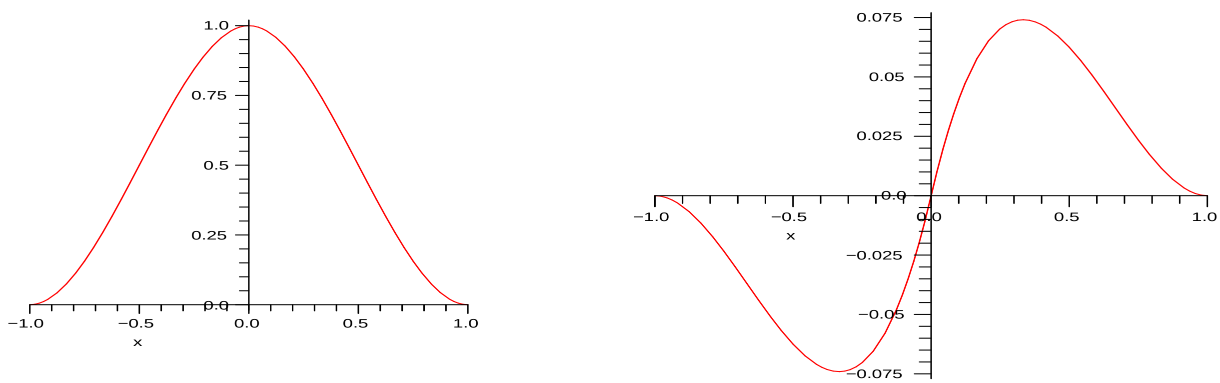

Figure 1. The functions $\phi(x)$ and $\psi(x)$