13.7. Strong, weak, absolute, and relative stability. To formalize the stability problem discussed above, we now define several concepts of stability that seek to differentiate between methods which exhibit numerical instability and those that do not. These definitions usually refer to the difference equations obtained by applying the multistep method to the model problem:

$$y' = \lambda y, \qquad y(x_0) = y_0,$$

whose exact solution is $y(x) = y_0 e^{\lambda(x-x_0)}$. In this case, the resulting difference equation is:

$$y_{n+1} = \sum_{i=0}^{p} a_i y_{n-i} + h \sum_{i=-1}^{p} b_i \lambda y_{n-i},$$

which may be rewritten as:

$$y_{n+1}[1 - h\lambda b_{-1}] = \sum_{i=0}^{p} [a_i + h\lambda b_i] y_{n-i}.$$

This is a linear constant coefficient difference equation. The associated characteristic polynomial is:

$$z^{p+1}[1 - h\lambda b_{-1}] = \sum_{i=0}^{p} [a_i + h\lambda b_i] z^{p-i}.$$

When $h = 0$, this becomes just $\rho(z) = 0$. In general, it is $\rho(z) - h\lambda\sigma(z) = 0$.

We previously defined zero-stability as requiring that all roots of $\rho(z)$ have modulus ≤ 1 and all roots of modulus one to be simple. Since we want our method to be consistent (necessary for convergence), $z = 1$ is always a root of $\rho(z) = 0$.

Definition: The roots of $\rho(z)$ of modulus one are called *essential roots*. The root $z = 1$ is called the *principal root*. The roots of $\rho(z)$ of modulus < 1 are called *nonessential roots*.

Definition: A linear multistep method is *strongly stable* if all roots of $\rho(z)$ are ≤ 1 in magnitude and only one root has magnitude one. If more than one root has magnitude one, the method is called *weakly or conditionally stable*. Note, we still require only simple roots of magnitude one. Also, note these definitions refer to the case $h = 0$.

Returning to the example $y_{n+1} = y_{n-1} + 2hf_n$, we have $\rho(z) = z^2 - 1$, so the roots are $z = \pm 1$. Hence, this is a weakly stable method. For the specific problem $y' = -y$, $y(0) = 1$, the roots of the difference equation were

$$z_1 = -h + \sqrt{1 + h^2}, \qquad z_2 = -h - \sqrt{1 + h^2}.$$

The problem was that since $|z_2| > 1$, the corresponding parasitic solution blew up. The basic idea of strong stability is that since the roots of a polynomial are continuous functions of the coefficients, for $h\lambda$ near zero, the roots of $\rho(z) - h\lambda\sigma(z) = 0$ are near the roots of $\rho(z) = 0$. If the method is strongly stable, all extraneous roots have magnitude < 1, so for $|h\lambda|$ small enough, all roots of $\rho(z) - h\lambda\sigma(z) = 0$ will also have magnitude < 1. Hence the parasitic solution corresponding to this root will decay as $n \to \infty$, instead of blowing up to ruin the approximate solution. Other definitions of stability try to be more precise in defining the values of $h\lambda$ for which the parasitic solutions remain bounded.

Definition: A linearly multistep method is said to be *absolutely stable* for those values of $h\lambda$ for which all roots r_s of $\pi(r, h\lambda) = \rho(r) - h\lambda\sigma(r) = 0$ satisfy $|r_s| \leq 1$ (and if $|r_s| = 1$, then r_s is simple).

In other words, all solutions of the test problem

$$y_{n+1}[1 - b_{-1}h\lambda] = \sum_{i=0}^{p}(a_i - h\lambda b_i)y_{n-i}$$

remain bounded as $n \to \infty$. If the method is absolutely stable for all $h\lambda \in (\alpha, \beta)$, the interval (α, β) is called the *interval of absolute stability.*

Example: midpoint rule $y_{n+1} = y_{n-1} + 2h\lambda y_n$. The characteristic polynomial is $r^2 - 2h\lambda r - 1 = 0$, so $r = h\lambda \pm \sqrt{h^2\lambda^2 + 1}$. Clearly if $h\lambda < 0$, then $|h\lambda - \sqrt{h^2\lambda^2 + 1}| > 1$ and if $h\lambda > 0$, then $|h\lambda + \sqrt{h^2\lambda^2 + 1}| > 1$. Hence, this method is only absolutely stable for $h\lambda = 0$, so there is no interval of absolute stability.

The definition of absolutely stable determines an interval in which parasitic solutions do not grow. However, if the true solution is increasing, i.e., $\lambda > 0$, then it is not a problem if parasitic solutions grow, provided they do not grow faster than the true solution.

Definition: A linear multistep method is said to be *relatively stable* for those values of $h\lambda$ for which all roots r_s of $\pi(r, h\lambda)$ satisfy $|r_s| \leq |r_0|$, and if $|r_s| = |r_0|$, then r_s is simple. Here r_0 is the principle root, i.e., the root with the property that $\lim_{h\to 0} r_0(h) = 1$. If the method is relatively stable for all $h\lambda \in (\alpha, \beta)$, the interval (α, β) is called the *interval of relative stability.*

Example: midpoint rule $r_0 = h\lambda + \sqrt{h^2\lambda^2 + 1}$, $r_1 = h\lambda - \sqrt{h^2\lambda^2 + 1}$. For relative stability, we require $|h\lambda - \sqrt{h^2\lambda^2 + 1}| \leq |h\lambda + \sqrt{h^2\lambda^2 + 1}|$, i.e., $h\lambda \geq 0$. So the interval of relative stability is $[0, \infty)$.

Remark: There are various similar definitions in the literature that make slight changes (e.g., using $<$ instead of $\leq$ and not requiring simple roots).

Note that the concept of relative stability does not apply to one-step methods since there is only one root of $\rho(r)$, but the concept of absolute stability does apply.

Example: Euler's method: $y_{n+1} = y_n + hf_n$. When $f(x, y) = \lambda y$, we get $y_{n+1} = y_n + h\lambda y_n$, so $r_0 = 1 + h\lambda$. For absolute stability, we need $-1 \leq 1 + h\lambda \leq 1$, i.e., $-2 \leq h\lambda \leq 0$. Hence, the interval of absolute stability is $[-2, 0]$.

Example: Trapezoidal rule $y_{n+1} = y_n + (h/2)(f_{n+1} + f_n)$. When $f(x, y) = \lambda y$, we get $y_{n+1} = y_n + (h\lambda/2)(y_{n+1} + y_n)$. Hence, $(1 - h\lambda/2)y_{n+1} = (1 + h\lambda/2)y_n$. So the only root of the characteristic polynomial is $r_0 = (1 + h\lambda/2)/(1 - h\lambda/2)$. For absolute stability, we need $|r_0| \leq 1$, i.e., $h\lambda \leq 0$.

This is the best one can obtain, since one can show that $h\lambda > 0$ cannot belong to the interval of absolute stability.

What do these definitions (referring to the model linear problem) tell us about solving the full nonlinear problem, i.e., the application of the $(p+1)$-step method

$$y_{n+1} = \sum_{i=0}^{p} a_i y_{n-i} + h \sum_{i=-1}^{p} b_i f_{n-i}$$

to the problem $y' = f(x, y)$, $y(x_0) = y_0$?

The true solution $y(x)$ satisfies

$$y(x_{n+1}) = \sum_{i=0}^{p} a_i y(x_{n-i}) + h \sum_{i=-1}^{p} b_i f(x_{n-i}, y(x_{n-i})) + T_n,$$

where $T_n = \mathcal{L}[y(x_n); h]$ is the local truncation error of the method. Letting $e_n = y(x_n) - y_n$, we get

$$e_{n+1} = \sum_{i=0}^{p} a_i e_{n-i} + h \sum_{i=-1}^{p} b_i [f(x_{n-i}, y(x_{n-i})) - f(x_{n-i}, y_{n-i})] + T_n.$$

Suppose $\partial f/\partial y$ exists and is continuous. Then, by the Mean Value Theorem, there exists ξ_{n-i} in the interval spanned by $y(x_{n-i})$ and y_{n-i} such that

$$f(x_{n-i}, y(x_{n-i})) - f(x_{n-i}, y_{n-i}) = \frac{\partial f}{\partial y}(x_{n-i}, \xi_{n-i}) e_{n-i}.$$

Hence,

$$\left[1 - hb_{-1}\frac{\partial f}{\partial y}(x_{n+1}, \xi_{n+1})\right] e_{n+1} = \sum_{i=0}^{p} \left[a_i + hb_i \frac{\partial f}{\partial y}(x_{n-i}, \xi_{n-i})\right] e_{n-i} + T_n.$$

If we look at this equation locally, and assume that $\partial f/\partial y$ and T_n do not vary much, then we expect the local behavior to be what we get by replacing $\partial f/\partial y$ by a constant λ and T_n by a constant T. Thus, we are led to look at the constant coefficient difference equation:

$$[1 - hb_{-1}\lambda]e_{n+1} = \sum_{i=0}^{p} [a_i + hb_i\lambda]e_{n-i} + T.$$

If the roots of the characteristic polynomial $(1-hb_{-1}\lambda)r^{p+1} = \sum_{i=0}^{p}[a_i+hb_i\lambda]r^{p-i}$ are distinct, then the general solution is $e_n = \sum_{s=0}^{p} d_s r_s^n$ plus a particular solution. It is easy to check that a particular solution is given by $-T/(h\lambda \sum_{i=-1}^{p} b_i)$ (just set $e_k = e$ for all k and use the fact that $\sum_{i=0}^{p} a_i = 1$). This is exactly the equation we have been studying. So λ represents a local approximation to $\partial f/\partial y$. For $\lambda > 0$, we restrict the step size h so that $h\lambda$ falls within the interval of relative stability, and for $\lambda < 0$, we restrict the step size h so that $h\lambda$ falls within the interval of absolute stability. We can also use these definitions to compare intervals of stability for different methods and use it as a criteria for selecting a desirable method.

Example: We have seen that the interval of absolute stability for Euler's method is $-2 \le h\lambda \le 0$. If we solve the model problem $y' = \lambda y$, $y(0) = 1$, then the true solution is $y(x) = e^{\lambda x}$. For $\lambda < 0$ $y(x)$ is decreasing. If we apply Euler's method to this problem, then

$y_{n+1} = y_n + h\lambda y_n = (1 + h\lambda)y_n$, so the approximate solution is $y_n = (1 + h\lambda)^n$. For $\lambda < 0$, unless we choose h so that $h\lambda > -2$, we will have $1 + h\lambda < -1$ and so $|y_n|$ will grow as we increase n. Thus, in addition to choosing h to control the local error, we must also choose h sufficiently small to control the stability of the approximation scheme. Note that in the case of the trapezoidal rule, since the interval of absolute stability is $h\lambda \leq 0$, there would be no restriction on h coming from stability.