13.5.3. *Convergence of multistep methods.* Definition: The linear multistep method defined by the formula

$$y_{n+1} = \sum_{i=0}^{p} a_i y_{n-i} + h \sum_{i=-1}^{p} b_i f_{n-i}, \tag{13.1}$$

is said to be convergent, if for all initial value problems $y' = f(x, y)$, $y(a) = \eta$, where f satisfies hypotheses (A) and (B) of Theorem 10 (i.e, f defined and continuous and satisfies a Lipschitz condition), we have that

$$(i) \qquad \lim_{\substack{h\to 0, n\to\infty \\ nh = x-a}} y_n^h = y(x_n)$$

holds for all $x \in [a, b]$ and all solutions $\{y_n^h\}$ of the difference equation (13.1) having starting values y_k^h, $k = 0, \ldots, p$ satisfying (ii) $\lim_{h\to 0} y_k^h = \eta$, $k = 0, \ldots, p$.

Note that this definition requires that (i) be satisfied not only for the sequence $\{y_n^h\}$ defined with exact starting values $y(a + kh)$ (for these (ii) is certainly satisfied), but also for all sequences whose starting values tend to the correct starting value η as $h \to 0$. This more stringent condition is imposed, since in practice it is almost never possible to start a computation with exact values.

Remark: To be a convergent method, the approximate solution must converge to the true solution for any problem of a certain class (i.e., f Lipschitz continuous in y). It is not enough to have convergence for a particular problem.

13.5.4. *Linear difference equations.* One of the methods for analyzing multistep methods for the approximation of ordinary differential equations involves the application of the method to the model problem $y' = \lambda y$, where λ is a constant. For this simple problem, the equation defining the numerical method becomes a linear difference equation. We now discuss the solution of such equations.

Definition: A difference equation is a relationship of the form $f(k, y_k, y_{k-1}, \ldots, y_{k-N}) = 0$ between an independent variable k and an unknown sequence of values $\{y_k\}$. A solution of a difference equation is a sequence of numbers $\{y_k\}$ that satisfies $f(k, y_k, y_{k-1}, \ldots, y_{k-N}) = 0$ for all k in some set I of consecutive integers.

Example: $f(k, y_k, y_{k-1}) = y_k - y_{k-1} - 1 = 0$. The solution is the sequence $\{y_k\}$, where $y_k = k + C$, with C constant and I is the set of all integers.

Example: $y_k = q y_{k-1}$. The solution is the sequence $\{y_k\}$, where $y_k = Cq^k$, where C is a constant. In both cases, we obtain a family of solutions depending on the paramter C. If we are given an initial condition such as $y_0 = A$ to determine C, then we have an initial value problem for the difference equation.

Definition: The order of a difference equation is the difference between the largest and smallest subscript (of y) appearing in the equation.

Definition: A linear difference equation is a difference equation of the form:

$$a_0(k)y_k + a_1(k)y_{k-1} + \cdots + a_N(k)y_{k-N} = b(k),$$

where $a_i(k)$ and $b(k)$ are functions only of k and do not depend on y.

Definition: A linear difference equation is called homogeneous if $b(k) \equiv 0$ for all k.

We now consider the solution of an Nth order linear homogeneous difference equation with constant coefficients. Since both a_0 and $a_N \neq 0$, we take $a_0 = 1$ (i.e., divide the equation by a_0 and relabel). Thus, the equation has the form

$$y_k + a_1 y_{k-1} + \cdots + a_N y_{k-N} = 0.$$

To solve this equation, we look for solutions of the form $y_k = z^k$, where z is a constant to be determined. Then z satisfies

$$z^k + a_1 z^{k-1} + \cdots + a_N z^{k-N} = 0, \quad \text{i.e.,}$$
$$z^{k-N}[z^N + a_1 z^{N-1} + \cdots + a_N] = 0.$$

This expression will be zero not only when $z = 0$ (the trivial solution), but also when z is a root of

$$\rho(z) = z^N + a_1 z^{N-1} + \cdots + a_N = 0.$$

$\rho(z)$ is called the characteristic polynomial of the difference equation.

Suppose we solve $\rho(z) = 0$ and find m distinct roots $z_1, \ldots, z_m$, with p_μ the multiplicity of z_μ. Then $z_\mu^k, kz_\mu^k, \ldots, k^{p_\mu - 1} z_\mu^k$ are also solutions of the difference equation. This gives us N solutions of the difference equation, which turn out to be linearly independent. The general solution of the difference equation is a linear combination of these solutions, i.e.,

$$y_k = \sum_{\mu=1}^{m} \sum_{j=1}^{p_\mu} C_{\mu j} k^{j-1} z_\mu^k.$$

Since we will assume that the coefficients of our difference equation are real, if z is a complex root of $\rho(z)$, the complex conjugate $\bar{z}$ is also a root of $\rho(z)$, with the same multiplicity, i.e., if $z = re^{i\theta}$ is a root, so is $\bar{z} = re^{-i\theta}$. Hence z^k and $\bar{z}^k$ are solutions of the difference equation. The part of the general solution of the difference equation corresponding to these solutions is $Az^k + B\bar{z}^k$, where A and B are complex. These may be written in terms of real solutions using the following formulas:

$$Az^k = Ar^k e^{i\theta k} = Ar^k[\cos(\theta k) + i\sin(\theta k)], \qquad B\bar{z}^k = Br^k e^{-i\theta k} = Br^k[\cos(\theta k) - i\sin(\theta k)],$$

where if $z = x + iy$ then $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$. Hence,

$$Az^k + B\bar{z}^k = r^k[(A + B)\cos(\theta k) + i(A - B)\sin(\theta k)]. = r^k[a\cos(\theta k) + b\sin(\theta k)],$$

where $a = A + B$ and $b = i(A - B)$. Thus, in the formula for the general solution, we can replace the linearly independent complex solutions z^k and $\bar{z}^k$ by the linearly independent real solutions $r^k\cos(\theta k)$ and $r^k\sin(\theta k)$.

We obtain the general solution of a linear nonhomogeneous difference equation by adding a particular solution to the general solution of the homogeneous difference equation. A particular solution can sometimes be found by the method of undetermined coefficients, i.e., by trying a solution of the form constant times the right hand side of the equation.

Example: Suppose an amount A is borrowed at an interest rate of i per payment period and must be paid back in n payments of equal amounts S. Determine S in terms of A, i, and n.

Let P_k be the principal outstanding after the kth payments. Then $P_0 = A$ and $P_n = 0$. Now

$$P_{k+1} = P_k + iP_k - S = (1+i)P_k - S.$$

The general solution of the homogenous equation is $a(1+i)^k$ for any constant a. Since the right hand side of the equation is constant, we look for a particular solution which is a constant, say b. Then $b = (1+i)b - S$, so $b = S/i$. Hence the general solution of the full difference equation is given by $P_k = a(1+i)^k + S/i$. Applying the initial condition $P_0 = A$, we find that $a = A - S/i$. Hence, $P_k = [A - S/i](1+i)^k + S/i$. Finally, we determine S from the condition that $P_n = 0$, i.e., $[A - S/i](1+i)^n + S/i = 0$. Hence,

$$S = \frac{Ai(1+i)^n}{(1+i)^n - 1} = A\frac{i}{1-(1+i)^{-n}}.$$

Theorem 12. *A necessary condition for the convergence of linear multistep method is that the method be consistent, i.e.,*

$$1 = \sum_{i=0}^{p} a_i, \qquad 1 = -\sum_{i=0}^{p} ia_i + \sum_{i=-1}^{p} b_i.$$

Proof. If the method is convergent, then it is convergent for the IVP $y' = 0$, $y(0) = 1$, whose exact solution is $y(x) = 1$. For this problem, the general linear multistep method becomes $y_{n+1} = \sum_{i=0}^{p} a_i y_{n-i}$. Let the starting values $y_0, \dots y_p$ be exact, i.e., equal to 1. Since the method is convergent, we must have that $y_n^h \to 1$ as $h \to 0$, $n \to \infty$, and $nh = x$. Hence, letting $n \to \infty$ in the expression $y_{n+1} = \sum_{i=0}^{p} a_i y_{n-i}$, we get $1 = \sum_{i=0}^{p} a_i$.

To establish the second equality, we consider the IVP $y' = 1$, $y(0) = 0$, whose exact solution is $y(x) = x$. The difference equation is now $y_{n+1} = \sum_{i=0}^{p} a_i y_{n-i} + h\sum_{i=-1}^{p} b_i$. Consider the sequence $y_n = nhA$, $n = 0, 1, \dots,$, where

$$A = \frac{\sum_{i=-1}^{p} b_i}{1 + \sum_{i=0}^{p} ia_i}.$$

We will first show that the sequence $\{y_n\}$ is a solution of the difference equation. To see this, we compute

$$\sum_{i=0}^{p} a_i y_{n-i} + h\sum_{i=-1}^{p} b_i = \sum_{i=0}^{p} a_i(n-i)hA + h\sum_{i=-1}^{p} b_i = \sum_{i=0}^{p} a_i(n-i)hA + hA(1 + \sum_{i=0}^{p} ia_i)$$

$$= hA + hAn\sum_{i=0}^{p} a_i = (n+1)hA = y_{n+1},$$

where we have used the first identity. We next observe that this sequence also satisfies the condition that $\lim_{h\to 0} y_n = 0$, $n = 1, 2, \ldots p$. Since the method is convergent, $y_n^h \to x$ as $h \to 0$, $n \to \infty$, and $nh = x$, i.e., $nhA = x$ for $nh = x$. Hence $A = 1$, so the second equality is established. $\square$