10.4. **Construction of Gaussian quadrature formulas.** Using these results, we now return to the problem of finding abscissas $x_0, \cdots, x_n$ and weights $H_0, \cdots, H_n$ so that

$$\int_a^b w(x)P(x)\,dx = \sum_{j=0}^n H_j P(x_j)$$

for all polynomials $P(x)$ of degree $\leq 2n+1$. We make use of the following result, which we have already established.

Lemma 4. *For any distinct points $x_0, \cdots, x_n$ with $a < x_i < b$, there exist unique constants $H_0, \cdots, H_n$, such that for any polynomial of degree $\leq n$,*

$$\int_a^b w(x)P(x)\,dx = \sum_{j=0}^n H_j P(x_j).$$

We note that the constants H_j are given by the formula

$$H_j = \int_a^b w(x)L_{j,n}(x)\,dx, \qquad \text{where} \qquad L_{j,n}(x) = \prod_{\substack{i=0 \\ i\neq j}}^n (x - x_i)/(x_j - x_i).$$

Our main result is the following theorem.

Theorem 8. *There exist abscissas $x_0, \cdots, x_n$ and weights $H_0, \cdots, H_n$ such that*

$$\int_a^b w(x)P(x)\,dx = \sum_{j=0}^n H_j P(x_j)$$

for all polynomials $P(x)$ of degree $\leq 2n+1$ if and only if the x_j are the zeroes of Φ_{n+1}.

Proof. For any $x_0, \ldots, x_n$, let $P_{n+1}(x) = \prod_{j=0}^n (x - x_j)$. Then any polynomial $P(x)$ of degree $\leq 2n+1$ can be written in the form $P(x) = Q(x)P_{n+1}(x) + R(x)$, where Q and R are polynomials of degree at most n. Then the quadrature formula

$$\int_a^b w(x)P(x)\,dx = \sum_{j=0}^n H_j P(x_j) \tag{10.1}$$

becomes

$$\begin{aligned} \int_a^b w(x)P(x)\,dx &= \int_a^b w(x)Q(x)P_{n+1}(x)\,dx + \int_a^b w(x)R(x)\,dx \\ &= \sum_{j=0}^n H_j Q(x_j)P_{n+1}(x_j) + \sum_{j=0}^n H_j R(x_j) = \sum_{j=0}^n H_j R(x_j), \end{aligned} \tag{10.2}$$

since $P_{n+1}(x_j) = 0$. Hence, the quadrature formula (10.1) will hold for any polynomial $P(x)$ of degree $\leq 2n+1$ if and only if the quadrature formula (10.2) holds for all $Q(x)$ and $R(x)$ of degree $\leq n$.

Now if the x_js are the zeroes of $\Phi_{n+1}(x)$, i.e., if $P_{n+1}(x) = \Phi_{n+1}(x)$, then

$$\int_a^b w(x)Q(x)P_{n+1}(x)\,dx = \int_a^b w(x)Q(x)\Phi_{n+1}(x)\,dx = 0$$

since $(\Phi_{n+1}, Q) = 0$ for all polynomials Q of degree $\leq n$. By Lemma 4, there exist constants $H_0, \ldots, H_n$ such that

$$\int_a^b w(x)R(x)\,dx = \sum_{j=0}^{n} H_j R(x_j).$$

Hence, (10.2) holds if the x_j are the zeroes of $\Phi_{n+1}(x)$.

Now suppose (10.2) holds for all $Q(x)$ and $R(x)$ of degree $\leq n$. Then it must hold when $R(x) \equiv 0$, i.e.,

$$\int_a^b w(x)Q(x)P_{n+1}(x)\,dx = (Q, P_{n+1}) = 0$$

for all Q of degree $\leq n$. By Lemma 3, $P_{n+1}(x) = c\Phi_{n+1}(x)$, $c \neq 0$ and so the x_js are the zeroes of $\Phi_{n+1}(x)$. □

We next derive a formula for the error in this approximation.

Theorem 9. *If the x_j and H_j are defined as in Lemma 4 and Theorem 8, and if $f(x) \in V$ satsifies $f^{(2n+2)}$ is continuous in (a, b), then*

$$E = \int_a^b w(x)f(x)\,dx - \sum_{j=0}^{n} H_j f(x_j) = \frac{\gamma_{n+1}}{(2n+2)!} f^{(2n+2)}(\xi)$$

for some $\xi \in (a, b)$.

Proof. Denote by $Q(x)$ the polynomial of degree $\leq 2n+1$ which solves the Hermite interpolation problem $Q(x_i) = f(x_i)$, $Q'(x_i) = f'(x_i), i = 0, \ldots, n$. By Theorem 8, the Gauss quadrature formula is exact for $Q(x)$, i.e.,

$$\int_a^b w(x)Q(x)\,dx = \sum_{j=0}^{n} H_j Q(x_j) = \sum_{j=0}^{n} H_j f(x_j).$$

Hence, by the error formula for polynomial interpolation,

$$\begin{aligned}
E &= \int_a^b w(x)f(x)\,dx - \sum_{j=0}^{n} H_j f(x_j) = \int_a^b w(x)[f(x) - Q(x)]\,dx \\
&= \int_a^b w(x) f[x_0, x_0, x_1, x_1, \cdots, x_n, x_n, x] \prod_{j=0}^{n} (x - x_j)^2\,dx \\
&= \frac{f^{(2n+2)}(\xi_x)}{(2n+2)!} \int_a^b w(x) \prod_{j=0}^{n} (x - x_j)^2\,dx \\
&= \frac{f^{(2n+2)}(\xi_x)}{(2n+2)!} \int_a^b w(x) \Phi_{n+1}^2(x)\,dx,
\end{aligned}$$

where we have used the fact that Φ_{n+1} is the unique polynomial of degree n with leading coefficient equal to one, with zeros at $x_0, \ldots, x_n$. □

Remark: It is possible to show that the H_j can be computed without integrations. The result is:

$$H_j = \frac{\gamma_n}{\Phi_n(x_j)\Phi'_{n+1}(x_j)}.$$

10.5. **Examples of orthogonal polynomials.** We next present standard sets of orthogonal polynomials corresponding to different choices of weight functions $w(x)$ and limits of integration a and b.

(i) $a = -1$, $b = 1$, $w(x) \equiv 1$. Legendre polynomials. The corresponding quadrature formula is called the Legendre-Gauss quadrature formula.

(ii) $a = 0$, $b = \infty$, $w(x) = e^{-x}$. Laguerre polynomials.

(iii) $a = -1$, $b = 1$, $w(x) = 1/\sqrt{1-x^2}$. Chebyshev polynomials.

(iv) $a = -\infty$, $b = \infty$, $w(x) = e^{-x^2}$. Hermite polynomials.

There are several advantanges to including a weight function $w(x)$. When either or both a and b are infinite, it is convenient to choose $w(x)$ to insure convergence of the integral of $w(x)f(x)$, where $f(x)$ is a polynomial of arbitrary degree (as in (ii) and (iii) above). In singular integrals, e.g., with terms like $1/\sqrt{1-x^2}$, it is convenient to have formulas and expressions for the error that do not depend on such terms.