1. Polynomial Interpolation

1.1. **Weierstrass Approximation Theorem.** If $f(x)$ is continuous on a finite interval $[a,b]$, then given $\epsilon > 0$, there exists n depending on ϵ and a polynomial $P_n(x)$ of degree $\leq n$ such that $|f(x) - P_n(x)| \leq \epsilon$ for all $x \in [a,b]$.

The proof uses Bernstein polynomials: These polynomials are defined on the interval $[0,1]$ by: $B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f(k/n)$. One can show that $\lim_{n\to\infty} B_n(x) = f(x)$ uniformly in $[0,1]$, so we can approximate continuous functions to any accuracy using polynomials. If $y \in [a,b]$, we can reduce the problem to the interval $[0,1]$ by the change of variable $x = (y-a)/(b-a)$. If $f(x)$ satisfies a Lipschitz condition, i.e., if

$$|f(x) - f(y)| \leq L|x-y|, \quad \text{for all } x, y \in [0,1],$$

then it is known that

$$|f(x) - B_n(x)| \leq \tfrac{1}{2} L n^{-1/2}.$$

1.2. **Polynomial interpolation:** Given $n+1$ distinct points $x_0, \dots, x_n$, and function values $f(x_0), \dots, f(x_n)$, find a polynomial $P_n(x)$ of degree $\leq n$ satisfying $P_n(x_j) = f(x_j), \quad j = 0, 1, \dots, n$.

We say P_n interpolates f at $x_0, \dots, x_n$ and consider two methods of constructing such a polynomial.

1.3. **Lagrange form of the interpolating polynomial.** : Define for $k = 0, 1, \dots, n$:

$$L_{k,n}(x) = \prod_{\substack{j=0 \\ j\neq k}}^{n} \frac{(x - x_j)}{(x_k - x_j)}, \quad n \geq 1, \qquad L_{0,0}(x) = 1.$$

Claim: $P(x) = \sum_{k=0}^{n} L_{k,n}(x) f(x_k)$ is a polynomial of degree $\leq n$ that interpolates f at $x_0, \dots, x_n$. This representation is called the Lagrange form of the interpolating polynomial.

Another way of thinking about this form of a polynomial is that we are writing $P(x) = \sum_{k=0}^{n} L_{k,n}(x) P(x_k)$. The functions $L_{k,n}(x)$ are fixed and the polynomial $P(x)$ is then uniquely determined by its values $P(x_k)$ at the $n+1$ points $x_0, \dots, x_n$. These values $P(x_0), \dots, P(x_n)$ are the degrees of freedom of $P(x)$ (in this form of the polynomial), i.e., they are the quantities we are free to choose that uniquely determine $P(x)$. Once we know we can write any polynomial in this form, it is then easy to solve the interpolation problem, i.e., we simply replace $P(x_k)$ by $f(x_k)$.

Observe that $L_{k,n}(x)$ is a product of n monomials in x and hence is a polynomial of degree $\leq n$. Taking linear combinations still gives a polynomial of degree $\leq n$.

Next: $L_{k,n}(x_i) = 0, i \neq k$. $L_{k,n}(x_i) = 1, i = k$. Hence, $P(x_i) = f(x_i)$.

Uniqueness: Suppose $Q(x)$ is another interpolating polynomial of degree $\leq n$. Then $R(x) = P(x) - Q(x)$ is of degree $\leq n$ and equals zero at $x_0, \dots, x_n$. Since $R(x) = 0$ at the distinct points $x_1, \dots, x_n$, $R(x)$ has the form $R(x) = A(x-x_1)\cdots(x-x_n)$ for some constant A. Then $R(x_0) = 0$ implies $A = 0$, so $R(x) = 0$.

Note: There exists other polynomials of degree $d > n$ interpolating f at $x_0, \dots, x_n$.

Examples: $n = 0$: $P_0(x) = f(x_0)$.

$n = 1$: $P_1(x) = \frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1)$.

1.4. **Newton form of the interpolating polynomial.** Note we can also write:

$$P_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0).$$

This is an example of the Newton form of the interpolating polynomial. In general, we want to write $P_n(x) = P_{n-1}(x) + Q_n(x)$, where P_{n-1} interpolates f at $x_0, \dots, x_{n-1}$ and $Q_n(x)$ has a simple form.

By the definitions of P_n and P_{n-1}, $Q_n(x)$ is of degree $\leq n$ and $= 0$ at $x_0, \dots, x_{n-1}$. Hence, $Q_n(x) = A_n(x - x_0)\cdots(x - x_{n-1})$ for some constant A_n. So we need only determine A_n.

Now $f(x_n) = P_n(x_n) = P_{n-1}(x_n) + Q_n(x_n)$. Hence,

$$\begin{aligned}
A_n &= \frac{f(x_n) - P_{n-1}(x_n)}{\prod_{j=0}^{n-1}(x_n - x_j)} \\
&= f(x_n) - \sum_{k=0}^{n-1}\left[\prod_{\substack{j=0 \\ j\neq k}}^{n-1} \frac{x_n - x_j}{x_k - x_j}\right] f(x_k) \Big/ \prod_{j=0}^{n-1}(x_n - x_j) \\
&= \frac{f(x_n)}{\prod_{j=0}^{n-1}(x_n - x_j)} - \sum_{k=0}^{n-1} \frac{f(x_k)}{\prod_{\substack{j=0 \\ j\neq k}}^{n-1}(x_k - x_j)} \cdot \frac{1}{x_n - x_k} \\
&= \sum_{k=0}^{n} \frac{f(x_k)}{\prod_{\substack{j=0 \\ j\neq k}}^{n}(x_k - x_j)}. \qquad (1.1)
\end{aligned}$$

We refer to A_n given by this formula as the n^{th} divided difference of f with respect to $x_0, \dots, x_n$ and denote it by $f[x_0, x_1, \dots, x_n]$.

Defining the divided difference $f[x_0] = f(x_0)$, we can generate the polynomials $P_n(x)$ recursively. Beginning with $P_0(x) = f(x_0) = f[x_0]$, we obtain

$$P_1(x) = P_0(x) + f[x_0, x_1](x - x_0) = f[x_0] + f[x_0, x_1](x - x_0).$$

Then

$$\begin{aligned}
P_2(x) &= P_1(x) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\
&= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1).
\end{aligned}$$

In general, we get the formula:

$$P_n(x) = \sum_{i=0}^{n} f[x_0, \ldots, x_i] \prod_{j=0}^{i-1} (x - x_j), \tag{1.2}$$

where we define $\prod_{j=0}^{-1}(x - x_j) = 1$.

Formula (1.2) is known as the Newton form of the interpolating polynomial.

In order to use formula (1.2), we must of course be able to evaluate the divided difference $f[x_0, \ldots, x_i]$. Using (1.1), we have that

$$f[x_0, x_1] = \frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

and

$$f[x_0, x_1, x_2] = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)}.$$

Observing that

$$\frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} = \frac{f(x_1)}{(x_1 - x_0)(x_0 - x_2)} - \frac{f(x_1)}{(x_1 - x_2)(x_0 - x_2)},$$

we can rewrite

$$f[x_0, x_1, x_2] = \left\{ \frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0} \right\} / (x_2 - x_0) = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}.$$

The reason for writing $f[x_0, x_1, x_2]$ in this form is that it indicates an easy way of generating divided differences recursively. We can show in general that

$$f[x_0, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}.$$

This formula allows us to generate all the divided differences needed for the Newton formula in a simple manner by using a divided difference table, rather than using formula (1.1). We illustrate such a table in the case $n = 4$.

TABLE 1

Divided difference table

x_k	$f(x_k)$	f[,]	f[,,]	f[,,,]	f[,,,,]
x_0	$f(x_0)$				
		$f[x_0, x_1]$			
x_1	$f(x_1)$		$f[x_0, x_1, x_2]$		
		$f[x_1, x_2]$		$f[x_0, x_1, x_2, x_3]$	
x_2	$f(x_2)$		$f[x_1, x_2, x_3]$		$f[x_0, x_1, x_2, x_3, x_4]$
		$f[x_2, x_3]$		$f[x_1, x_2, x_3, x_4]$	
x_3	$f(x_3)$		$f[x_2, x_3, x_4]$		
		$f[x_3, x_4]$			
x_4	$f(x_4)$				

The divided differences in the table are calculated a column at a time using the formula

$$f[x_i, \dots, x_{i+k}] = \frac{f[x_{i+1}, \dots, x_{i+k}] - f[x_i, \dots, x_{i+k-1}]}{x_{i+k} - x_i}.$$

The coefficients needed for the Newton formula are then found at the beginning of each column.

Observe how entries are added to the table each time a a new data point is added. For example, if we started with the entries in the table involving only the points x_0, x_1, x_2, x_3, and added the point x_4, we would successively generate $f[x_3, x_4]$. $f[x_2, x_3, x_4]$, $f[x_1, x_2, x_3, x_4]$, and finally $f[x_0, x_1, x_2, x_3, x_4]$, the additional divided difference needed for the construction of $P_4(x)$.

We shall return frequently to the idea of degrees of freedom of a function f. These are quantities that uniquely determine the function f. In all our applications, these will be values of f or its derivatives at specific points, or possibly moments of f, i.e., quantities of the form $\int_a^b x^r f(x)\,dx$ for some integers $f \geq 0$. Note that a function may be uniquely determined by several sets of degrees of freedom, and the choice of which ones to use and how to represent the function will depend on the application. For example, we can also represent any polynomial of degree $\leq n$ by its Taylor series expansion about a point x_0, i.e.,

$$P_n(x) = \sum_{j=0}^{n} \frac{P^{(j)}(x_0)}{j!}(x - x_0)^j.$$

In this representation, we see that $P_n(x)$ is uniquely determined by the quantities $P^{(j)}(x_0)$, i.e., its derivatives up to order n at the point x_0.