Notes on the Vibrating String Problem

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1 Derivation of the equations of motion

Consider a string on mass density ρ units of mass per unit length with the ends fixed a distance of L units of length apart. Let T denote the tension in the string. Think of a violin string, for example – stretched quite tight so that the tension is relatively high, and when the string vibrates, its oscillations do not have a large amplitude.

Let x denote the position along the string, and let $h(x)$ denote vertical the displacement (in the x, y plane) of the part of the string at distance x from the left fixed point at $x = 0$. When the string is at rest, $h(x) = 0$ for all x.

If the string is plucked, or otherwise set in motion, what will that motion be? We can deduce a system of equations for this motion from Newton's Second Law. Pick a large number N, and define

$$x_j = \frac{jL}{N} \qquad \text{and} \qquad \Delta x = \frac{L}{N} \ .$$

Then for $j = 0, \ldots, N$, x_j denote the position of the the center of a segment of the string of length Δx, and therefore of mass $\rho \Delta x$. Let $y_j(t)$ denote the vertical displacement (from the rest position) of the center of this segment at time t. Note that since the ends are fixed,

$$y_0(t) = y_N(t) = 0 \qquad \text{for all} \quad t \ . \tag{1.1}$$

For $1 \leq j \leq N-1$, the vertical acceleration of the jth segment is $y_j''(t)$ and so by Newton's Second Law,

$$(\rho \Delta x) y_j''(t) = F_j(t) \tag{1.2}$$

where $F_j(t)$ is the vertical component of the force acting on the jth segment at time t. The forces would be gravity and the tension in the string. But for a string under high tension, like a violin string, the tension is the only significant force. We therefore neglect gravity.

The segment at x_j is tugged on from the right and from the left. At time t, the force from the left is

$$T \frac{1}{\sqrt{\Delta x^2 + (y_{j-1}(t)) - y_j(t))^2}} \left((x_{j_1}, y_{j-1}(t)) - (x_j, y_j(t)) \right) \ .$$

Likewise, the force from the left is

$$T \frac{1}{\sqrt{\Delta x^2 + (y_{j+1}(t)) - y_j(t))^2}} \left((x_{j+1}, y_{j+1}(t)) - (x_j, y_j(t)) \right) \ .$$

Thus,

$$F_j(t) = T\frac{1}{\sqrt{\Delta x^2 + (y_{j-1}(t)) - y_j(t))^2}}\left(y_{j-1}(t) - y_j(t)\right) + \\ T\frac{1}{\sqrt{\Delta x^2 + (y_{j+1}(t)) - y_j(t))^2}}\left(y_{j+1}(t) - y_j(t)\right)\ . \quad (1.3)$$

We now make a crucial approximation: We assume that, since the tension is high, the amplitude of the vibrations is very small, so that for each j, $|y_{j+1}(t)) - y_j(t)|$ is small compared with Δx. Thus, in (1.3) we make the replacements

$$\sqrt{\Delta x^2 + (y_{j+1}(t)) - y_j(t))^2} \to \Delta x \qquad \text{and} \qquad \sqrt{\Delta x^2 + (y_{j+1}(t)) - y_j(t))^2} \to \Delta x\ .$$

Thus we approximate

$$F_j(t) \to \frac{T}{\Delta x}(y_{j+1}(t) - 2y_j(t) + y_{j-1}(t))\ .$$

Using this in (1.2), we obtain the system

$$(\rho\Delta x)y_j''(t) = \frac{T}{\Delta x}(y_{j+1}(t) - 2y_j(t) + y_{j-1}(t))\ . \quad (1.4)$$

for $j = 1, \dots, N_1$, and where $x_0(t) = x_N(t) = 0$ for all t since the ends of the string are fixed.

Now let K denote the $(N-1) \times (N-1)$ matrix given by

$$K_{i,j} = \frac{1}{\Delta x^2}\begin{cases} 2 & i = j \\ -1 & i = j-1 \\ -1 & i = j+1 \end{cases}\ .$$

Also, define

$$c := \sqrt{\frac{T}{\rho}}\ . \quad (1.5)$$

Then, introducing the vecotor $\mathbf{y}(t) = (y_1(t), \dots, y_{N-1}t))$, we can rewrite (1.4) as

$$\mathbf{y}''(t) = -c^2 K\mathbf{y}(t)\ . \quad (1.6)$$

This is a familiar equation, since the matrix K is not only symmetric; all of its eigenvalues are positive. To see this, make the simple computation showing that for any $\mathbf{x} \in \mathbb{R}^{N-1}$.

$$\mathbf{x} \cdot K\mathbf{x} = x_1^2 + x_{N-1}^2 + \sum_{j=1}^{N-2}(x_{j+1} - x_j)^2\ .$$

Consequently, if $K\mathbf{x} = \mu\mathbf{x}$, and $\mathbf{x} \neq \mathbf{0}$,

$$\mu\|\mathbf{x}\|^2 = \mathbf{x} \cdot K\mathbf{x} > 0\ .$$

Hence, all of the eigenvalues μ of K are strictly positive. we could proceed to analyze this equation by seeking the eigenvectors of K. However, it is in many ways more enlightening to make a further approximation, taking the *continuum limit*.

Let $h(x,t)$ denote the vertical displacement of the part of the string at horizontal coordinate x and at time t. Form the vector

$$\mathbf{h}(t) = (h(x_1,t),\ldots,h(x_{N-1},t))\ .$$

Then for each $j = 1,\ldots,N-1$,

$$(K\mathbf{h}(t))_j = \frac{-h(x_{j-1},t) + 2h(x_j,t) - h(x_{j+1},t)}{\Delta x^2}\ ,$$

where $x_0 = 0$ and $x_N = L$. (Recall that $h(0,t) = h(L,t) = 0$ for all t.) This can be rewritten as

$$(K\mathbf{h}(t))_j = -\frac{1}{\Delta x}\left(\frac{h(x_j+\Delta x,t) - h(x_j,t)}{\Delta x} - \frac{h(x_j,t) - h(x_j-\Delta x,t)}{\Delta x}\right)\ .$$

If for each t, $h(x,t)$ is twice continuously differentiable as a function of x, then

$$\lim_{\Delta x\to 0}\frac{1}{\Delta x}\left(\frac{h(x_j+\Delta x,t) - h(x_j,t)}{\Delta x} - \frac{h(x_j,t) - h(x_j-\Delta x,t)}{\Delta x}\right) = \frac{\partial^2}{\partial x^2}h(x_j,t)\ .$$

Also, by the definition of $\mathbf{y}(t)$,

$$\mathbf{y}_j''(t) = \frac{\partial^2}{\partial t^2}h(x_j,t)\ .$$

The continuum approximation to (1.6) then is to take Δx to zero, and hence N to infinity, and to replace (1.6) with

$$\frac{\partial^2}{\partial x^2}h(x,t) = c^2\frac{\partial^2}{\partial x^2}h(x,t)\ . \tag{1.7}$$

where the equation (1.7) is to hold at *each* $x \in (0,L)$, and for all t. We also require that $h(x,t)$ satisfy the *boundary conditions*

$$h(0,t) = 0 = h(L,t) \qquad \text{for all} \qquad t\ . \tag{1.8}$$

The equation (1.7) is known as the *wave equation.*

2 Solving the wave equation

Let $g(x)$ be any twice continuously differentiable function on $\mathbb{R}$. Consider the function $h(x,t)$ defined by

$$h_+(x,t) = g(x-ct)\ .$$

Then

$$\frac{\partial^2}{\partial t^2}h_+(x,t) = c^2g''(x-ct) \qquad \text{and} \qquad \frac{\partial^2}{\partial x^2}h_+(x,t) = 2g''(x-ct)\ .$$

Thus, $h_+(x,t)$ solves (1.7). Likewise, defining

$$h_-(x,t) = g(x+ct)\ .$$

Using these two solutions, we can solve the wave equation (1.7) subject to the boundary conditions (1.8) with the initial data

$$h(x,0) = h_0(x) \qquad \text{and} \qquad \frac{\partial}{\partial t}h(0,x) = 0 \qquad \text{for all} \quad x \in (0,L)\ . \tag{2.1}$$

Here is how: Let g be any function on $\mathbb{R}$ such that

$$g(x) = h_0(x) \qquad \text{for all} \quad x \in (0, L) \ .$$

Then the function

$$h(x,t) = \frac{1}{2}(g(x - ct) + g(x + ct))$$

satisfies the wave equation (1.7) and moreover, for $x \in (0, L)$, for which $g(x) = h_0(x)$,

$$h(x, 0) = \frac{1}{2}(g(x) + g(x)) = h_0(x) \ ,$$

while

$$\frac{\partial}{\partial t} h(0, x) = \frac{1}{2}(cg'(x) - cg'(x)) = 0 \ .$$

Hence the function $h(x,t)$ satisfies both the equation (1.7) and the initial data (2.1) *no matter how we extend the function* h_0 *to the whole of* $\mathbb{R}$. However, if we do this in an arbitrary way, the boundary conditions (1.8) will *not* be satisfied, and the resulting function will *not* describe the motion of a string that is fixed at $x = 0$ and $x = L$.

To satisfy the boundary conditions, we must extend h_0 to all of $\mathbb{R}$ in a very particular way. There is a *unique* extension of h_0 to all of $\mathbb{R}$ that is antisymmetric about both $x = 0$ and $x = L$. One may produce this function by repeated reflections, as described in class. Let $\overline{h_0}$ be this antisymmetric extension. Then for each t and each $x \in (0, L)$,

$$\overline{h_0}(-ct) = -\overline{h_0}(ct) \ ,$$

and so

$$\frac{1}{2}(\overline{h_0}(-ct) + \overline{h_0}(ct)) = 0 \ .$$

Likewise, for each t and each $x \in (0, L)$,

$$\overline{h_0}(L - ct) = -\overline{h_0}(L + ct) \ ,$$

and so

$$\frac{1}{2}(\overline{h_0}(L - ct) + \overline{h_0}(L + ct)) = 0 \ .$$

Therefore

$$h(x,t) = \frac{1}{2}(\overline{h_0}(x - ct) + \overline{h_0}(x + ct)) \tag{2.2}$$

satisfies the wave equation (1.7), the boundary conditions (1.8) and the initial conditions (2.1).

In the next section we show that this solution is the *only* such solution: Thus, the boundary conditions and the initial data together completely determine the solution.

3 Uniqueness

We know that there is exactly one solution of (1.6) for given initial data $\mathbf{y}(0)$ and $\mathbf{y}'(0)$. One way to see this is to use the energy function

$$E(t) = \|\mathbf{y}'(t)\|^2 + c^2\mathbf{y} \cdot K\mathbf{y} \ .$$

Then, since K is a symmetric matrix so that

$$\mathbf{y} \cdot K\mathbf{y}'(t) = K\mathbf{y}(t) \cdot \mathbf{y}'(t) \ ,$$

and since the dot product is commutative,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E(t) &= 2\mathbf{y}'(t) \cdot \mathbf{y}''(t) + c^2 2\mathbf{y}'(t) \cdot K\mathbf{y}(t) \\ &= 2\mathbf{y}'(t) \cdot [\mathbf{y}''(t) + c^2 K\mathbf{y}(t)] = 0 \ . \end{aligned}$$

where in the last equality we used (1.6).

Now, suppose $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ are two solutions of (1.6) with $\mathbf{y}_1(0) = \mathbf{y}_2(0)$ and $\mathbf{y}_1'(0) = \mathbf{y}_2'(0)$. Define $\mathbf{z}(t) = \mathbf{y}_2(t) - \mathbf{y}_1(t)$. Then since the equation (1.6) is linear, $\mathbf{z}(t)$ solves (1.6). Also, for this solution $\mathbf{z}(t)$, $E(0) = 0$, and so $E(t) = 0$ for all t. But as we have seen above, if $\mathbf{z}(t) \neq \mathbf{0}$, then $\mathbf{z}(t) \cdot K\mathbf{z}(t) > 0$, and obviously $\|\mathbf{z}'(t)\|^2 \geq 0$. Hence the fact that $E(t) = 0$ for all t implies that $\mathbf{z}(t) = \mathbf{0}$ for all t, and hence $\mathbf{y}_2(t) = \mathbf{y}_1(t)$ for all t.

Let us now adapt this argument to the continuum limit. Given a solution $h(x,t)$ of (1.7), define the energy

$$E(t) = \int_0^L \left|\frac{\partial}{\partial t}h(x,t)\right|^2 \mathrm{d}x + c^2 \int_0^L \left|\frac{\partial}{\partial x}h(x,t)\right|^2 \mathrm{d}x \ .$$

Differentiating under the integral sign,

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = 2\int_0^L \frac{\partial}{\partial t}h(x,t)\frac{\partial^2}{\partial t^2}h(x,t)\mathrm{d}x + 2c^2 \int_0^L \frac{\partial}{\partial x}h(x,t)\frac{\partial^2}{\partial x \partial t}h(x,t)\mathrm{d}x \ .$$

Now, integrating by parts,

$$\int_0^L \frac{\partial}{\partial x}h(x,t)\frac{\partial^2}{\partial x \partial t}h(x,t)\mathrm{d}x = \frac{\partial}{\partial x}h(x,t)\frac{\partial}{\partial t}h(x,t)\bigg|_0^L - \int_0^L \frac{\partial}{\partial t}h(x,t)\frac{\partial^2}{\partial x^2 \partial t}h(x,t)\mathrm{d}x \ .$$

If we further suppose that h satisfies the boundary conditions (1.8), we have also that

$$\frac{\partial}{\partial t}h(0,t) = 0 = \frac{\partial}{\partial t}h(L,t) \qquad \text{for all} \qquad t \ . \tag{3.1}$$

Then the boundary terms in the integration by parts are zero, and

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = 2\int_0^L \frac{\partial}{\partial t}h(x,t)\left[\frac{\partial^2}{\partial t^2}h(x,t) - c^2\frac{\partial^2}{\partial c^2}h(x,t)\right] \mathrm{d}x = 0 \ .$$

Thus, the energy is constant along solution of the wave equation (1.6) that also satisfy the boundary conditions (1.8). Hence if $h_1(x,t)$ and $h_2(x,t)$ are two such solutions with

$$h_1(x,0) = h_2(x,0) \qquad \text{and} \qquad \frac{\partial}{\partial t}h_1(x,0) = \frac{\partial}{\partial t}h_2(x,0) \qquad \text{for all} \quad x \in (0,L) \ ,$$

then $h(x,t) := h_1(x,t) - h_2(x,t)$ is another such solution for which the initial energy $E(0)$ is zero. Therefore, for this solution $h(x,t)$, $E(t) = 0$ for all t. But this means that $h(x,t) = 0$ for all t and all $0 < x < L$. Hence $h_2(x,t) = h_1(x,t)$ for all t and all $0 < x < L$. This proves the uniqueness.

Notice that the boundary conditions played a crucial role in this proof – as they must: We have seen that there are many solutions to the initial data problem if one drops the boundary conditions.

4 Taking into account a non-zero initial velocity

Let us now try to solve the wave equation (1.6) subject to the boundary conditions (1.8) and also subject to the initial conditions

$$h(x,0) = 0 \qquad \text{and} \qquad \frac{\partial}{\partial t}h(0,x) = v_0(x) \qquad \text{for all} \quad x \in (0,L) \ . \tag{4.1}$$

Let $w(x) = \int_0^x v_0(z)\mathrm{d}z$ for $0 \leq x \leq L$. Let $u(x)$ be *any* extension of u to all of $\mathbb{R}$. Define

$$h(x,t) = \frac{1}{2c}(u(x+ct) - u(x-ct)) \ . \tag{4.2}$$

Then

$$h(x,0) = \frac{1}{2c}(u(x) - u(x)) = 0 \ ,$$

and, for all $x \in (0,L)$,

$$\frac{\partial}{\partial t}h(x,0) = \frac{1}{2c}(cu'(x) + cu'(x)) = v_0(x)0 \ .$$

Therefore, (4.2) gives us a solution of (1.6) and (4.1). However, it will not generally satisfy the boundary conditions. To do this, let $\overline{w}$ denote the unique extension of w from $(0,L)$ to all of $\mathbb{R}$ that is symmetric about both $x = 0$ and $x = L$. Then

$$\frac{1}{2}(\overline{w}(ct) - \overline{w}(-ct)) = 0 \qquad \text{and} \qquad \frac{1}{2}(\overline{w}(L+ct) - \overline{w}(L-ct)) = 0$$

for all t. Hence

$$h(x,t) = \frac{1}{2c}(\overline{w}(x+ct) - \overline{w}(x-ct)) \ . \tag{4.3}$$

is the solution of (1.6) subject to both (1.7) and (4.1).

Finally, taking advantage of the linearity of the wave equation, we can solve (1.6) subject to (1.7) and the general boundary conditions

$$h(x,0) = h_0(x) \qquad \text{and} \qquad \frac{\partial}{\partial t}h(0,x) = v_0(x) \qquad \text{for all} \quad x \in (0,L) \ . \tag{4.4}$$

by adding the soiutions (2.2) and (4.3) to the special case studied above.